The Swirling Vortex

J. Serrin

Email alerting service	Receive free email alerts when new articles cite this article - sign up in the box at the top right-hand corner of the article or click **here**

[325]

THE SWIRLING VORTEX

By J. SERRIN
University of Minnesota

(*Communicated by L. Howarth, F.R.S.—Received* 9 *December* 1969—*Revised* 27 *July* 1971)

[Plates 2 to 4]

CONTENTS

We consider an infinite vortex line in a viscous fluid interacting with a plane boundary surface at right angles to the line. If the boundary surface were absent, the vortex would impart to the fluid a circular motion about the vortex line with speed inversely proportional to the distance to the line. The presence of the boundary surface, however, leads to a secondary flow due to the forced adherence of the fluid at the surface.

The purpose of the paper is to describe a family of exact solutions of the Navier–Stokes equations which applies to the above situation. Under quite general hypotheses, it is shown that there can exist only three types of motion compatible with the assumed structure. In the first kind, the radial velocity component (using spherical polar coordinates about the point where the vortex meets the plane surface) is directed inward along the plane surface and upward along the axis of the vortex. In the second type of motion the radial velocity component is directed inward along the plane surface and downward on the axis, with a compensating outflow at an intermediate angle. In the third kind the radial velocity is directed outward near the plane and downward on the central axis. The results can also be used as a basis for numerical calculations of the solutions in question, and several typical flow patterns have been explicitly computed in order to illustrate the theory.

The paper concludes with a discussion of the relation between the theoretical solutions and observed phenomena near the point of contact of tornadoes with the ground; this requires that the flows under discussion be considered as mean motions in a turbulent flow with constant eddy viscosity. The present work adds theoretical weight to the argument that central downdrafts can occur in tornadoes. Moreover, the model provides an explanation, other than centrifugal action, for the frequent appearance of a cascade effect at the foot of both tornadoes and water-spouts; finally it offers a unified point of view from which to consider the diversity of flow patterns observed when vortex fields interact with a boundary surface.

Introduction

Consider an infinite vortex line in a viscous fluid or gas, interacting with a plane boundary surface at right angles to the line. If the boundary surface were absent, the vortex would impart to the fluid a circular velocity about the vortex line of magnitude C/r, where r is the radial distance to the line and the line is assumed for simplicity to be straight. On the other hand, the presence of the boundary surface, which we may suppose to be horizontal and represented by the equation $z = 0$ in rectangular coordinates, leads to a secondary flow due to the forced aherence of the fluid at the surface.

In a general way, under these circumstances we may expect that the loss of circulatory velocity near the plane surface, due to adherence, will produce an unbalanced pressure field near this surface, which falls as one moves towards the axis of the vortex. As a consequence the secondary flow should be directed toward the vortex, and this in turn should create an updraft along its axis. Such a picture is an intriguing one, especially if one is interested in explaining various phenomena of dynamic meteorology. An important and fundamental difficulty arises, however, in attempting to find specific mechanisms (i.e. particular solutions of the Navier–Stokes equations) which account for such swirling or spiral vortex motions in the space $z > 0$.

In this regard, the Russian engineer M. A. Goldstik discovered an interesting exact solution of the Navier–Stokes equations for viscous incompressible flow. His solution partially displays the features required but unfortunately is available only at values of the dimensionless parameter

$$k = C/2\nu$$

so low as to cast doubt on its applicability to many situations of practical significance (here ν denotes the kinematic viscosity). It turns out, however, that Goldstik's solution is just one of a broader class, and that in this broader class we can find swirling motions which exist at arbitrary values of k. Naturally, such vortex motions cannot be entirely transparent and simple to the eye (otherwise surely someone would have long since discovered their structure). Thus it might be expected that the initial picture of a simple secondary motion involving inflow and an updraft needs further modification, and indeed this is the case for the solutions we shall discuss here.

Specifically, we find that there exist three types of swirling vortex motion compatible with the structure assumed here (see § 1). In the first kind, the radial velocity component (in spherical coordinates) is directed inward near the plane $z = 0$ and upward near the axis of the vortex; thus in executing a motion of this kind the fluid near the boundary plane is drawn inward in tightening spirals toward the z-axis and is then swirled upwards in helical paths about the axis. This type of motion, similar to that determined by Goldstik and displaying features of the kind indicated in the opening discussion, can exist however only for values of k less than 2.86. In the second kind of motion the velocity component is directed outward near $z = 0$ and downward near the vortex axis. This type of flow exists at arbitrary values of k, as does also the third kind in which the radial velocity is directed inward both near the plane $z = 0$ and near the vortex filament. For the latter motion there is of course a compensating outflow occurring at a suitable intermediate cone of directions. The downdraft in the second and third types of motion tends to be fairly strong as is the compensating outflow in the third case. Moreover, in the second type of motion, namely when there is a downdraft along the vortex line and an outflow near $z = 0$, the flow pattern develops a boundary layer as $C/2\nu$ becomes large. That is, except in the immediate neighbourhood of the boundary surface the angular velocity about the vortex line becomes

nearly C/r, as required by the irrotational flow, while in the boundary layer itself the angular velocity rapidly changes from zero at the boundary surface to the main streaming value C/r required by the vortex.

Clearly the variety and complexity of these flows may be of value in explaining the diverse behaviour of naturally occurring vortex phenomena. At the same time, irrespective of such applications, the motions studied here are also interesting for the fact that they are among the more complicated examples in which a boundary value problem for the Navier–Stokes equations can be reduced to the solution of ordinary differential equations (and hence for which a relatively explicit determination of the flow pattern is possible). Considering the difficulty always experienced in finding significant exact solutions of the Navier–Stokes equations, there seems to be sufficient reason on this ground alone to justify the present analysis.

This much being said, I feel compelled to add that the solutions discussed here are not intended as a definitive and final answer to the problem of interaction of a vortex filament with a plane boundary. There is certainly no reason to suppose that some other (possibly non-steady) type of flow might not equally well or better describe this interaction; moreover, the singularity existing along the axis of the line vortex is itself a mathematical idealization of the phenomenon of vortex motion in real fluids. Finally if one is concerned with meteorological phenomena it must surely be decided at some stage whether the Navier–Stokes equations themselves are suitable for the explanation of these phenomena, or alternately whether thermal and compressibility effects play a significant part in the actual motion (besides supplying the ultimate driving energy). On the available evidence, however, there seems to be no conviction that naturally occurring vortices must necessarily and in all cases be fundamentally compressible phenomena. Indeed, the time scales at which tornado phenomena occur do not indicate the action of either buoyancy or compressibility as an important determinant of the local structure of the air motion; and even in cases where compressibility, buoyancy, and other effects are surely present, it is not improbable that further understanding can be gained from an incompressible analogy. Consequently, at the present stage of investigation it remains a reasonable assumption to adopt the Navier–Stokes equations as an appropriate model for fluid motion in a vortex.

Before proceeding to the main considerations, it should perhaps be added that when one is dealing with a solution of the Navier–Stokes equations the actual energy source is irrelevant to the *structure* of the solution: the solution exists as a dynamically possible fluid motion and nothing more is required of it. In the real world the solution expresses a local situation, and the energy to produce and drive it comes from outside. The mode by which this energy is imparted to the flow is of course ultimately of interest; when we deal as here with a solution which idealizes a swirling motion by means of a line vortex, the vortex singularity can serve as a source of momentum, this being entirely consistent with the interpretation of the solution as an asymptotic limit (as the core radii shrinks to zero) of a family of flows not containing a singularity. Morton (1966, p. 186) gives an excellent account both of the relation of the concentrated vortex core to the main motion and of the driving mechanisms likely to be involved in typical tornado and waterspout phenomena.

The paper is divided into four parts. The first chapter is concerned with the basic structure of the vortex and with the derivation of an appropriate system of differential equations governing the motion. The system itself consists of a coupled pair of nonlinear ordinary differential equations, one of the fourth order and the other of second order, subject to five boundary conditions. The value $k = C/2\nu$ appears as a parameter multiplying the highest order derivatives; thus, taking into account that there is one fewer boundary condition than the order of the system

allows, it is apparent that we must ultimately deal with a two-parameter family of solutions. Using a technique introduced by Goldstik and in fact appearing in a special form even earlier in the work of Slezkin, the basic equations can be reduced to a first-order integro-differential equation of essentially more manageable proportions. This reduction is carried out in § 2. The second chapter is then concerned with the structure of the solutions of the problem; in particular, it is shown there that solutions have the general form described above.

Chapter III is a general study of the existence and non-existence of motions corresponding to different values of k. Because of the inherently complicated nature of the basic equations this work involves some difficulties. We have therefore made every effort to proceed as simply as possible, while at the same time aiming at completeness. The final section of chapter III contains the results of numerical calculations carried out at typical values of k.

With the exception of the work of Goldstik, most theoretical models of the interaction of a vortex and a plane surface have not included a singularity at the vortex axis. In the work of Burgers (1948) and Sullivan (1959), maintenance of a continuous velocity field throughout the vortex core is, however, accomplished at the cost of dropping the adherence condition at the boundary surface; moreover, in this work the velocity field does not approach zero as one proceeds radially away from the vortex (in fact, the speed becomes arbitrarily large), a drawback which remains even following boundary-layer analysis of the transition from slip flow to adherence at the boundary plane. Unsteady vortices corresponding to those of Burgers and Sullivan have been studied in interesting papers of Rott (1958), and Bellamy-Knights (1970). In work more closely allied to the present approach, Long (1958, 1961) has carried out a similarity analysis analogous to that given in § 1, and has considered boundary conditions which avoid the singularity at the axis, though again at the expense of dropping the adherence condition at the boundary surface.

It would be quite removed from our purposes to review the literature on boundary layers induced by vortex motion. We do however draw attention to a recent paper of Burggraf, Stewartson & Belcher (1971) in which they present a boundary layer analysis of a potential vortex interacting with a finite disk. In their work it is shown that a finite mass flux occurs at $r = 0$, and they comment that this should cause an eruption at the vortex axis. It is precisely here that a further alteration of their flow field would be necessary; our work bears this out in that we do not find steady motions which exhibit inflow-type boundary layers, but obtain instead the rather more complicated flow patterns described earlier. Further valuable work on the boundary layer induced by a maintained vortex is due to Kuo (1971).

We emphasize that our model obeys both the strict adherence condition at the boundary surface and the requirement that the velocity approach zero as one proceeds radially away from the vortex. On the other hand, because of the singularity at the axis, the physical interpretation of the results must be approached with care. The final part of the paper summarizes our results and contains a discussion of their possible bearing on meteorological phenomena. In particular, the present model adds theoretical weight to the argument (advanced by Oerstad in 1838 and periodically recurring in the literature) that central downdrafts can occur in tornadoes. Moreover, the model provides an explanation, other than centrifugal action, for the frequent appearance of a cascade effect at the foot of both tornadoes and waterspouts; finally it offers a unified point of view from which to consider the diversity of flow patterns observed when vortex fields interact with a boundary surface.

Chapter I. Formulation of the problem

1. *Basic equations*

It is convenient to introduce spherical polar coordinates (R, α, θ), where R is radial distance from the origin, α is the angle between the radius vector and the positive z-axis, and θ is the meridian angle about the z-axis. The positive z-axis is then described by $\alpha = 0$, the boundary plane $z = 0$ by $\alpha = \frac{1}{2}\pi$, and the half space $z > 0$ by $R > 0$, $0 \leqslant \alpha < \frac{1}{2}\pi$. The respective physical components of the velocity vector $\boldsymbol{v}$ in this coordinate system will be denoted by

$$v_R, \quad v_\alpha, \quad v_\theta. \tag{1}$$

We shall consider here steady-state fluid motions having the basic structure

$$v_R = \frac{G(x)}{r}, \quad v_\alpha = \frac{F(x)}{r}, \quad v_\theta = \frac{\Omega(x)}{r} \tag{2}$$

where $r = R \sin \alpha$ is the distance to the z-axis, and $x = \cos \alpha$. One finds immediately from the equation of continuity that

$$G = F' \sin \alpha,$$

the prime denoting differention with respect to x. The functions F and Ω are of course to be determined so that the flow is dynamically allowable (that is, satisfies the Navier–Stokes equations).

We remark that the particular functions $F \equiv 0$, $\Omega \equiv C$ yield a *line vortex* in space. Our purpose is to find related functions F and Ω appropriate to a swirling vortex motion adhering to the plane $z = 0$, as described in the introduction.

The first necessity is to derive appropriate differential conditions under which the basic motion (2) satisfies the Navier–Stokes equations. These equations, expressed in spherical polar coordinates, are reproduced in several standard fluid mechanics texts, cf., for example, Berker (1963, p. 6). Using the relation

$$\frac{\partial}{\partial \alpha} = -\sin \alpha \frac{\partial}{\partial x},$$

we find after some calculation that the R, α, and θ components of the Navier–Stokes equations become respectively

$$-FF'' - F'^2 - (F^2 + \Omega^2)\operatorname{cosec}^2 \alpha = -\frac{R^3}{\rho}\frac{\partial p}{\partial R} + \nu(F''' \sin^2 \alpha - 2F'' \cos \alpha),$$

$$-FF' - (F^2 + \Omega^2)\cot \alpha \operatorname{cosec} \alpha = -\frac{R^2 \sin \alpha}{\rho}\frac{\partial p}{\partial \alpha} - \nu F'' \sin^2 \alpha,$$

$$-F\Omega' = -\frac{R^2}{\rho}\frac{\partial p}{\partial \theta} + \nu \Omega'' \sin^2 \alpha,$$

where ρ is the density, p the pressure, and ν the kinematic viscosity, and where external forces have been absorbed into the pressure term.

From the last of these equations it is evident that p must be independent of θ (more precisely, p is at most linear in θ: since p must be periodic in θ, it is thus independent of θ). Now using the first equation, it follows that

$$\frac{p}{\rho} = \frac{A(x)}{R^2} + B(x).$$

Substituting this into the second equation shows that $B(x)$ must be a constant; hence writing $A(x) = \pi(x) \operatorname{cosec}^2 \alpha$ we obtain

$$\frac{p}{\rho} = \frac{\pi(x)}{R^2 \sin^2 \alpha} + \text{constant} = \frac{\pi(x)}{r^2} + \text{constant.} \tag{3}$$

Comparing this with the first equation then gives the pressure formula

$$-2\pi = F^2 + \Omega^2 + \{FF'' + F'^2 + \nu(F''' \sin^2 \alpha - 2F'' \cos \alpha)\} \sin^2 \alpha. \tag{4}$$

With the help of (4) we may eliminate p from the second equation. After some calculation this yields† the following ordinary differential system for the functions F and Ω,

$$\begin{aligned} \nu(1-x^2)\, F^{\mathrm{iv}} - 4\nu x F''' + FF''' + 3F'F'' &= -2\Omega\Omega'/(1-x^2) \\ \nu(1-x^2)\, \Omega'' + F\Omega' &= 0. \end{aligned} \tag{5}$$

Solutions of the system (5) provide a broad class of motions satisfying the Navier–Stokes equations. Our interest here will be in the particular boundary value problem obtained when the θ-component of the velocity approaches the vortex value C/r as α tends to zero and the adherence condition $\boldsymbol{v} = 0$ is satisfied when $\alpha = \frac{1}{2}\pi$. Thus we shall consider (5) on the interval $0 \leqslant x < 1$ with boundary conditions

$$\begin{aligned} \Omega = F = F' = 0 \quad &\text{when} \quad x = 0 \\ \Omega \to C \quad\quad\;\; &\text{as} \quad\;\; x \to 1 \end{aligned} \Bigg\}. \tag{6}$$

To these conditions must be added a restriction assuring that the vortex line $\alpha = 0$ is neither a source nor a sink of the fluid motion, namely

$$F \to 0 \quad \text{as} \quad x \to 1. \tag{7}$$

In order to verify the appropriateness of condition (7), it is convenient to introduce the function $\psi = RF(x)$. Let $\mathscr{S}$ be a surface of revolution about the z-axis generated by a meridian curve $\mathscr{C}$. Then using the fact that

$$v_R = -\frac{1}{R^2 \sin \alpha} \frac{\partial \psi}{\partial \alpha}, \quad v_\alpha = \frac{1}{R \sin \alpha} \frac{\partial \psi}{\partial R},$$

it is easy to show that the total flux of fluid through $\mathscr{S}$ is $2\pi |\psi_2 - \psi_1|$, where ψ_1 and ψ_2 denote the respective values of ψ at the end-points of the curve $\mathscr{C}$. If we now suppose that $\mathscr{C}$ joins two distinct points on the z-axis, the condition that this axis be neither a source nor a sink of the fluid motion is obviously that the flux through the corresponding closed surface $\mathscr{S}$ be zero. It follows that in this case $\psi_1 = \psi_2$, which in turn holds if and only if $F \to 0$ as $x \to 1$. This proves (7).

For later use, we note that the surfaces $\psi = \text{constant}$ contain the streamlines of the fluid motion. Thus these surfaces provide a useful method of flow visualization, in the same way as the streamfunction of an ordinary two-dimensional fluid motion.

Remarks. In §4 we shall show that the reduced pressure $\pi = r^2 p/\rho$ approaches the value $-\frac{1}{2}C^2$ when α tends to zero, in analogy with the case of a line vortex.

The basic separation of variables assumption can be clarified by a simple dimensional analysis. In the interaction problem as posed in the opening paragraphs of the introduction, neither a

---

† The derivation is facilitated by first writing π in the form

$$-2\pi = F^2 + \Omega^2 + \nu(1-x^2)\,S$$

where S denotes the expression enclosed by braces in (4). Equations (5) can also be derived from the last two relations on page 115 of Goldstein's *Modern developments in fluid mechanics*, vol. I (this process does not, however, yield formula (4) for the pressure).

preferred velocity nor a preferred distance is present: we have only the angular momentum C and the kinematic viscosity ν as basic physical constants, both having the dimensions L^2/T. Their ratio is therefore a dimensionless parameter k of the flow. This being the case, if we consider solutions which are independent of time and which are rotationally symmetric (i.e. the velocity components are independent of θ), then necessarily the dimensionless quantity $r\nu/C$ can depend only on the dimensionless variables α and k, unless we are prepared to consider solutions with some preferred distance added in. Since there is no *a priori* reason for including such a distance, it is therefore reasonable to consider solutions of the form (2).

Long (1958) introduced a system equivalent to (5) and gave an asymptotic analysis of the solutions under a set of boundary conditions different from those here. The system (5) also appears in Goldstik's work, though in somewhat disguised form (we note finally that the special case $\Omega \equiv 0$ of (5) was discovered by Slezkin; cf. Berker (1963, p. 59)).

2. *Reduction to an integro-differential equation*

In the preceding section we derived for the swirling vortex motion a sixth-order system of ordinary differential equations subject to five boundary conditions. It is therefore apparent that, even when ν and C are fixed, we must deal with a one-parameter family of fluid motions. This extra degree of freedom is vital to our further considerations.

The system (5) can be considerably simplified, since the first equation can be immediately integrated three times to give the relation

$$2\nu(1-x^2)F' + 4\nu xF + F^2 = -\int_0^x dx \int_0^x dx \int_0^x \frac{4\Omega\Omega'}{1-x^2}dx + Px^2 + Qx + S, \tag{8}$$

where P, Q, S are constants of integration.† Setting $x = 0$ and using the boundary conditions (6) yields $S = 0$.

In order to derive a further relation between the constants P and Q, note that the triple integral in (8) can be rewritten as

$$2\int_0^x \frac{(x-t)^2}{1-t^2}\Omega\Omega'\,dt;$$

this in turn can be integrated by parts to the form

$$2\int_0^x \frac{(x-t)(1-xt)}{(1-t^2)^2}\Omega^2\,dt$$

by making use of the boundary condition $\Omega(0) = 0$. The function Ω is bounded on the interval $[0, 1]$. Hence as x tends to 1 the above integral converges to the value

$$2\int_0^1 \frac{\Omega^2\,dt}{(1+t)^2}.$$

Consequently using (7) and (8) we find that

$$(1-x)F' \to \text{constant} \quad \text{as} \quad x \to 1. \tag{9}$$

If the constant is non-zero, say positive, then for x suitably near 1 we have $F' \geq \epsilon/(1-x)$. This implies in turn that F becomes infinite as x tends to 1, in contradiction to (7). Since the same argument applies if the constant is assumed to be negative, it follows that $(1-x)F' \to 0$ as $x \to 1$. Consequently from (8) we have

$$P + Q = 2\int_0^1 \frac{\Omega^2\,dt}{(1+t)^2}. \tag{10}$$

† For the special case $\Omega \equiv 0$ this integration was found by Slezkin.

One of the constants P and Q can be assigned arbitrarily, according to the remarks at the beginning of the section. Choosing P as the basic parameter, and using (10) to eliminate Q from (8), then gives the relation

$$2\nu(1-x^2)\,F' + 4\nu xF + F^2 = G(x), \tag{11}$$

where

$$G(x) = 2(1-x)^2 \int_0^x \frac{t\Omega^2}{(1-t^2)^2}\,\mathrm{d}t + 2x\int_x^1 \frac{\Omega^2\,\mathrm{d}t}{(1+t)^2} - P(x-x^2). \tag{12}$$

A final change of variables

$$F = 2\nu(1-x^2)f, \quad k = 1/2\nu \tag{13}$$

yields the integro-differential system

$$\left.\begin{aligned} f' + f^2 &= k^2\frac{G(x)}{(1-x^2)^2}\\ \Omega'' + 2f\Omega' &= 0 \end{aligned}\right\} \quad (0 \leqslant x < 1), \tag{14}$$

where $G(x)$ is given by (12), and f and Ω are subject to the boundary conditions

$$f = \Omega = 0 \quad\text{when}\quad x = 0, \qquad \Omega \to C \quad\text{as}\quad x \to 1. \tag{15}$$

In view of the change of variables $\Omega \to C\tilde{\Omega}$, $P \to C^2\tilde{P}$, $k \to \tilde{k}/|C|$, we can suppose without loss of generality that

$$C = 1. \tag{16}$$

This normalization will be maintained throughout the rest of the paper. Note that k then becomes the basic dimensionless parameter of the flow (i.e. the normalized value $\tilde{k}$ is $|C|/2\nu$). We note that in laminar flow k may be considered as a *Reynolds number*; if turbulent motion is assumed and ν correspondingly denotes the kinematic eddy viscosity, then it is more proper to view k as a self-regulating parameter indicating the basic level of turbulence present. Typical values of k are given in the final section of the paper.

So far, it has been shown that (14), (15) is a consequence of (5), (6), (7). In §4 it will be proved conversely that any solution of (14), (15) is also a solution of (5), (6), (7), so that the two systems are in fact equivalent.

For the special case $P = 1$ the system (14) was derived by Goldstik, by a much more complicated and circuitous procedure.

CHAPTER II. STRUCTURE OF SOLUTIONS

In this chapter we consider the general qualitative behaviour of solutions of the system (14), (15). The fundamental question of existence of solutions will be taken up in the following chapter.

3. *The function $G(x)$*

As a first step in discussing the structure of solutions, it is convenient to consider the behaviour of the function $G(x)$.

LEMMA 1. *The angular momentum function $\Omega(x)$ increases monotonically from 0 to 1 as x increases from 0 to 1.*

Proof. From the second equation of (14) we find

$$\Omega' = \text{const.} \exp\left(-2\int_0^x f \, dx\right). \tag{17}$$

Thus Ω' always has the same sign. The required conclusion therefore follows immediately from the given boundary conditions. For later use we note that (17) can be integrated once more to yield an explicit formula for Ω in terms of f.

LEMMA 2. *$G(x)$ satisfies the conditions*

$$G = 0, \quad G' = 2\int_0^1 \frac{\Omega^2 \, dt}{(1+t)^2} - P \quad when \quad x = 0;$$

$$G = 0, \quad G' = P - 1 \qquad\qquad when \quad x = 1.$$

Moreover, $G'''(x) < 0$ for $0 < x < 1$.

Proof. We have

$$G(x) = 2(1-x)^2 \int_0^x \frac{t\Omega^2}{(1-t^2)^2} \, dt + 2x \int_x^1 \frac{\Omega^2 \, dt}{(1+t)^2} - P(x - x^2),$$

$$G'(x) = -4(1-x) \int_0^x \frac{t\Omega^2}{(1-t^2)^2} \, dt + 2\int_x^1 \frac{\Omega^2 \, dt}{(1+t)^2} - P + 2Px.$$

The values of G and G' at $x = 0$ are apparent by inspection. Their values at $x = 1$ follow by application of L'Hôpital's rule for the indeterminate form $0 . \infty$. We have also

$$G''(x) = 4\int_0^x \frac{t\Omega^2}{(1-t^2)^2} \, dt - \frac{2\Omega^2}{1-x^2} + 2P,$$

$$G'''(x) = -\frac{4\Omega\Omega'}{1-x^2},$$

and the remaining part of the lemma follows at once.

In what follows it is convenient to put $G'(0) = Q$, in agreement with the relation (10). Then the following lemma holds.

LEMMA 3. *Let $Q = G'(0)$. Then for $0 < x < 1$ we have*

$$Q(x - x^2) < G(x) < (1 - P)(x - x^2).$$

Proof. The function $H(x) = G(x) - Q(x - x^2)$ satisfies $H(0) = H(1) = 0, H'(0) = 0$, and $H''' < 0$. From the last condition it follows that H'' has at most a single zero. Hence H is either convex, or first convex and then concave, or everywhere concave in the interval $0 < x < 1$.

The boundary conditions show that H cannot be everywhere convex or concave. In the remaining case (first convex and then concave) the boundary conditions imply without difficulty that $H > 0$. This proves the left hand inequality.

To prove the right hand inequality, consider the function $\hat{H}(x) = G(x) - (1-P)(x - x^2)$. Here $\hat{H}(0) = \hat{H}(1) = \hat{H}'(1) = 0$, and $\hat{H}''' < 0$. The dual of the previous demonstration now yields $\hat{H} < 0$, completing the proof.

As a consequence of lemma 3 it follows that, for $0 < x < 1$,

$$G(x) > 0 \quad \text{when} \quad Q \geqslant 0, \quad \text{and} \quad G(x) < 0 \quad \text{when} \quad P \geqslant 1.$$

Moreover, in the remaining case when $Q < 0$ and $P < 1$, the function G is first negative, has a single zero on the interval $0 < x < 1$, and is positive thereafter. In fact, $G''' < 0$ so that G'' has at most one zero. Consequently G is either convex, or concave, or first convex and then concave. Evidently only the last case is compatible with the boundary conditions $G(0) = G(1) = 0$ and $G'(0), G'(1) < 0$. But then G must be first negative and then positive.

4. *Behaviour of solutions*

In this section we investigate the behaviour of solutions of the system (14), (15). The purpose of the first theorem is to show that this system is equivalent to the original problem posed in § 1. We begin with three important lemmas.

LEMMA 4. *Suppose $P < 1$. Then* $\quad f \leqslant \frac{1}{4}(1-P)\, k^2 \ln \dfrac{1}{1-x}.$

Furthermore, if $Q \geqslant 0$ then f is positive, while if $Q < 0$ then f is either everywhere negative, or else first negative and then positive thereafter.†

Proof. By lemma 3 we have $G(x) \leqslant (1-P)(x - x^2)$. Thus using (14) and the fact that

$$x \leqslant \tfrac{1}{4}(1+x)^2,$$

we find $\quad\quad\quad\quad\quad f' \leqslant \tfrac{1}{4}(1-P)\, k^2/(1-x).$

Integrating this and making use of the initial condition $f(0) = 0$ proves the first part of the lemma.

Multiplying the first equation of (14) by the integrating factor $\exp\left(\int_0^x f\, dx\right)$ and carrying out the resulting integration yields the relation

$$f = k^2 \int_0^x \frac{G(t)}{(1-t^2)^2} \exp\left(-\int_t^x f\, du\right) dt,$$

where we have used the initial condition $f(0) = 0$. Now if $Q \geqslant 0$, then $G > 0$. Hence f is positive. When $Q < 0$ and $P < 1$ we have seen that G is first negative and then positive. Therefore f is negative for small x. At the first zero of f (if any) we surely have $G > 0$. But then $G > 0$ for the remainder of the interval, and $f > 0$ there exactly as before.

LEMMA 5. *If $P \geqslant 1$ then f is negative and decreasing.*

† It is convenient to exclude the points $x = 0, 1$ when referring to the sign of f. *This convention, and a similar one for G and Ω, will be adhered to throughout the paper.*

Proof. Recalling that $G < 0$ in the present case, we have

$$f' = k^2 \frac{G(x)}{(1-x^2)^2} - f^2 < 0, \tag{18}$$

and the required conclusion follows at once using the fact that $f(0) = 0$.

LEMMA 6. *Let f be a solution of* (14), (15). *Then*

$$f = O\left(\ln\frac{1}{1-x}\right) \quad as \quad x \to 1.$$

This result is slightly technical, and accordingly its proof will be deferred until the end of the chapter. We may now turn to the main results of the section.

THEOREM 1. *Let f, Ω be a solution of the system* (14), (15). *Then F, Ω is a solution of* (5) *satisfying the boundary conditions* (6), (7). *Moreover the reduced pressure $\pi(x)$ satisfies*

$$\pi(0) = -\tfrac{1}{2}P, \quad \pi(1) = -\tfrac{1}{2}. \tag{19}$$

Here we have assumed the normalization $C = 1$; otherwise the relations (19) would be replaced by $\pi(0) = -\tfrac{1}{2}PC^2$ and $\pi(1) = -\tfrac{1}{2}C^2$.

Proof of Theorem 1. In view of the derivation of (14), the result is obvious except possibly for the boundary conditions $F'(0) = F(1) = 0$ and the pressure relations (19). We have by (13), however,

$$F' = 2\nu(1-x^2)f' - 4\nu xf.$$

Since $f(0) = G(0) = 0$ it follows from (14) that $f'(0) = 0$. Consequently $F'(0) = 0$. Also by lemma 6

$$F = 2\nu(1-x^2)f \to 0 \quad as \quad x \to 1.$$

To prove (19), differentiate (11) twice and use the resulting relation to eliminate the quantity F''' from (4). This yields the formula

$$-2\pi = F^2 + \Omega^2 + \tfrac{1}{2}(1-x^2)(G'' - 4\nu F').$$

A further simplification is available if we use the identity $\tfrac{1}{2}(1-x^2)G'' + xG' - G = P - \Omega^2$; thus finally

$$-2\pi = P + 2F^2 + (4\nu F - G')x.$$

The required result now follows from lemma 2 and the boundary conditions for F.

The first condition of (19) provides a physical interpretation for the parameter P. Indeed, using (3) we find that

$$\frac{p}{\rho} = \begin{cases} -\dfrac{P}{2r^2} + \text{constant} & \text{when} \quad \alpha \sim 0 \quad (z\text{-axis}), \\[2mm] -\dfrac{1}{2r^2} + \text{constant} & \text{when} \quad \alpha = \tfrac{1}{2}\pi \quad (z = 0). \end{cases}$$

or, without the normalization $C = 1$,

$$\frac{p}{\rho} = \begin{cases} -\dfrac{PC^2}{2r^2} + \text{constant} & \text{when} \quad \alpha \sim 0 \\[2mm] -\dfrac{C^2}{2r^2} + \text{constant} & \text{when} \quad \alpha = \tfrac{1}{2}\pi. \end{cases}$$

Thus at the boundary plane the pressure p is (up to an additive constant) exactly P times the pressure in the corresponding free vortex. In particular if $P > 0$ then the pressure at the boundary

plane decreases as one approaches the vortex line (as noted heuristically in the introduction), though the rate of decrease will be the same as in a free vortex only when $P = 1$.

Theorem 1 shows that the problem (5), (6), (7) is equivalent to the problem (14), (15). Therefore from here on we may confine our discussion to the latter system. The following theorem is a more precise version of the results of lemmas 1, 4, 5 and 6.

THEOREM 2. *Let f, Ω be a solution of the system* (14), (15). *Then the following results hold.*

(i) *Suppose $P < 1$. If $G'(0) \geqslant 0$ then f is positive, and Ω is increasing and concave. If $G'(0) < 0$ then f is first negative, has a single zero on the interval $0 < x < 1$, and is positive thereafter; correspondingly Ω is first convex and then concave.*

(ii) *Suppose $P \geqslant 1$. Then f is negative and decreasing, and Ω is increasing and convex.*

(iii) *When $P = 1$ the function f tends to a finite negative limit as x tends to 1. In all other cases*

$$f \sim \tfrac{1}{4}(1-P)\, k^2 \ln \frac{1}{1-x}.$$

Proof. The first statement of part (i) is simply a rewording of part of lemma 4, together with the observation that $\Omega'' = -2f\Omega' < 0$. To prove the second part of (i), note that from lemma 4 either f is everywhere negative or else first negative and then positive thereafter. By part (iii), which we shall prove below, it is clear that $f > 0$ for x sufficiently near 1. Consequently the only case which can occur is that f is first negative, and then positive thereafter.

Part (ii) follows from lemma 5 and the fact that $\Omega'' = -2f\Omega' > 0$.

It remains only to prove part (iii). Suppose first that $P \neq 1$. Then by lemma 2 and lemma 6

$$(1-x)f' = k^2 \frac{G(x)}{(1-x)\,(1+x)^2} - (1-x)f^2 \to \tfrac{1}{4}(1-P)\, k^2,$$

as x tends to 1. Consequently f approaches either $+\infty$ or $-\infty$ and L'Hôpital's rule is applicable: thus

$$\lim \frac{f}{\ln 1/(1-x)} = \lim (1-x)f' = \tfrac{1}{4}(1-P)\, k^2,$$

as required.

Suppose next that $P = 1$. We assert that the quantity $G/(1-x)^2$ is integrable. Indeed, since $f = O(\ln 1/(1-x))$ it is clear from (17) that Ω' has a finite positive limit B as x tends to 1. Thus by the last formula in the proof of lemma 2

$$\lim (1-x)\, G''' = -2B.$$

By repeated application of L'Hôpital's rule, therefore,

$$\lim \frac{G}{(1-x)^2 \ln \dfrac{1}{1-x}} = \lim \frac{G'}{(1-x)\left(1 - 2\ln \dfrac{1}{1-x}\right)} = \lim \frac{G''}{2\ln \dfrac{1}{1-x} - 3} = \tfrac{1}{2}\lim (1-x)\, G''' = -B.$$

Consequently when $P = 1$ we have $G \sim -B(1-x)^2 \ln 1/(1-x)$ and the assertion is proved.

Having shown this, it follows with the help of lemma 6 and (18) that f' is negative and integrable. Hence f tends to a finite limit as x tends to 1. This completes the proof of the theorem.

The results of theorem 2 can be used to give a qualitative picture of the flow pattern corresponding to various parameter values. For simplicity we shall confine ourselves to the radial component of the velocity, though the other components can be treated equally well. Now by (2), (13) and (14) have

$$Rv_R = F' = 2\nu(1-x^2)f' - 4\nu x f = k\frac{G(x)}{1-x^2} - \frac{(1-x^2)f^2}{k} - \frac{2xf}{k}.$$

For small values of x the first term clearly dominates, and the sign of Rv_R will correspondingly agree with that of G. Moreover, as x tends to 1 we have with the help of lemma 2

$$Rv_R \sim -\tfrac{1}{2}(1-P)\,k\ln\frac{1}{1-x}$$

when $P \neq 1$, while F' tends to a *positive* limit if $P = 1$. In case $P < 1$ and $Q < 0$, the function f has a zero at some point $x = a$. Thus $G(a) > 0$ according to the proof of lemma 4. Therefore at $x = a$

$$Rv_R = k\frac{G(a)}{1-a^2} > 0.$$

Summarizing, we have the following results.

Let f be a solution of (14), (15). Then f is either everywhere positive, everywhere negative, or first negative and then positive. Moreover:

A. *For positive solutions f, the radial velocity is outward near the plane $z = 0$ and downward near the z-axis.*

B. *For solutions f which are first negative and then positive, the radial velocity is inward near the plane $z = 0$ and downward near the z-axis. This general motion toward the origin is balanced by a compensating outflow near the streamcone $\alpha_0 = \cos^{-1} a$.*

C. *For negative solutions f, the radial velocity is inward near the plane $z = 0$ and upward near the z-axis (we shall see later that such solutions can exist only at small values of the parameter k).*

Typical stream surfaces for these cases are illustrated in § 10. Using theorem 2 we can also obtain some idea of the relative magnitude of the secondary flow in comparison with the free vortex. Writing

$$\boldsymbol{v} = \boldsymbol{V} + \boldsymbol{w},$$

where $\boldsymbol{V}$ has components $(0, 0, 1/r)$, it is easy to check that the quantities $rw_R, rw_\alpha, rw_\theta$ are independent of R and tend uniformly to zero as α tends to zero, indicating clearly that the major effect near the vortex line is precisely the free vortex motion. For positive values of α, both $\boldsymbol{v}$ and $\boldsymbol{V}$ tend uniformly to zero as R tends to infinity. Finally, while the energy of the free vortex is infinite in any hemisphere $R \leqslant R_0$, $z \geqslant 0$, it can be shown that the energy of the secondary flow is finite and strictly proportional to R_0.

5. *Boundary-layer behaviour of positive solutions*

When f is positive, solutions of (14), (15) exhibit a boundary-layer behaviour, in which as k becomes large the function $\Omega(x)$ uniformly approaches 1 on compact subsets of $(0, 1]$. That is, for suitably large k the velocity component v_θ is arbitrarily near that of a free vortex, except for a thin boundary layer near the plane $z = 0$. The purpose of this section is to prove this and other related results.

Let $Q = G'(0)$. By lemma 3 $\qquad G(x) \geqslant Q(x - x^2).$ $\hfill (20)$

Here $Q \geqslant 0$ since we are supposing that f is positive (we shall in fact consider only *fixed positive* values of Q; the case $Q = 0$ requires a more delicate analysis, which is omitted here). Now using (14) and (20) it is clear that

$$f' \geqslant Qk^2\frac{x}{(1-x)\,(1+x)^2} - f^2.$$

Setting $c = Qk^2$ for simplicity in printing, and noting that $(1-x)(1+x)^2 < \frac{4}{3}$ for all x in question, we have then

$$f' \geqslant \tfrac{3}{4}cx - f^2. \tag{21}$$

Thus $f \geqslant \sigma$, where σ is defined by the differential equation

$$\sigma' = \tfrac{3}{4}cx - \sigma^2, \quad \sigma(0) = 0.$$

[The argument is as follows: put $g = f - \sigma$. Then

$$g' \geqslant \sigma^2 - f^2 = -(\sigma + f)\,g,$$

that is $g' + (\sigma + f)\,g \geqslant 0$. Multiplying by an integrating factor and carrying out the integration then yields (since $g(0) = 0$)

$$g \exp\left(\int_0^x (\sigma + f)\,\mathrm{d}x\right) \geqslant 0$$

as required. A similar comparison method will be used repeatedly in what follows, generally without specific reference.]

The function σ is obviously positive. Moreover, $\sigma' \leqslant \tfrac{3}{4}cx$, so that $\sigma \leqslant \tfrac{3}{8}cx^2$. Therefore in turn

$$\sigma' \geqslant \tfrac{3}{4}cx - \tfrac{9}{64}c^2x^4,$$

For $0 \leqslant x \leqslant c^{-\frac{1}{3}}$ this yields $\sigma' \geqslant \tfrac{1}{2}cx$ and $\sigma \geqslant \tfrac{1}{4}cx^2$. It is apparent that σ is an increasing function for $0 < x < 1$.† Thus for $x \geqslant c^{-\frac{1}{3}}$ there results

$$f \geqslant \sigma \geqslant \tfrac{1}{4}c^{\frac{1}{3}}. \tag{22}$$

Let $\delta = c^{-\frac{1}{3}} \doteq (Qk^2)^{-\frac{1}{3}}$. By (17)

$$\Omega'(x) = \Omega'(\delta) \exp\left(-2\int_\delta^x f\,\mathrm{d}x\right).$$

Since Ω is concave in the present case (see theorem 2) and $0 \leqslant \Omega \leqslant 1$, it is clear geometrically that $\Omega'(\delta) \leqslant 1/\delta$. Hence, making use of (22), we find that for $x \geqslant \delta$

$$\Omega'(x) \leqslant \frac{1}{\delta}\exp\left(\frac{\delta - x}{2\delta}\right) \leqslant \frac{2}{\delta}\exp\left(-\frac{x}{2\delta}\right).$$

Integrating backward from $x = 1$ (where $\Omega = 1$) then gives

$$|\Omega - 1| \leqslant 4\,\mathrm{e}^{-x/2\delta} \quad (x \geqslant \delta).$$

In particular, if $x \geqslant N\delta$ and $N \geqslant 1$, then

$$|\Omega - 1| \leqslant 4\,\mathrm{e}^{-\frac{1}{2}N}. \tag{23}$$

This establishes the existence of a boundary layer of nominal thickness $\delta \approx k^{-\frac{2}{3}} \approx \nu^{\frac{2}{3}}$ corresponding to fixed $Q > 0$.

Within the boundary layer the angular velocity component v_α is small, but the radial component v_R is positive and of order $k^{\frac{1}{3}}$. In order to establish this rigorously, we first show that for fixed $Q > 0$,

$$P \to 1 - Q \quad \text{as} \quad k \to \infty. \tag{24}$$

In fact, by lemma 2

$$1 - P - Q = 2\int_0^1 \frac{1 - \Omega^2}{(1 + t)^2}\,\mathrm{d}t,$$

so that by (23)

$$0 < \tfrac{1}{4}(1 - P - Q) \leqslant \int_0^1 \frac{1 - \Omega}{(1 + t)^2}\,\mathrm{d}t = \int_0^{N\delta} + \int_{N\delta}^1 < N\delta + 2\,\mathrm{e}^{-\frac{1}{2}N}$$

† The function $\tau = \tfrac{1}{2}\sqrt{(3cx)}$ satisfies $\tau' > \tfrac{3}{4}cx - \tau^2$, whence $\tau > \sigma$ and $\sigma' = \tau^2 - \sigma^2 > 0$.

valid for $1 \leqslant N \leqslant 1/\delta$, Now choose $N = 2 \ln 1/\delta$, which yields

$$0 < 1 - P - Q < 24\delta \ln 1/\delta \tag{25}$$

for $\delta \leqslant 1/\sqrt{e}$, i.e. for $Qk^2 \geqslant e^{\frac{3}{2}}$. The required limit behaviour now follows at once.

By lemma 3 we have $G(x) \leqslant (1-P)(x-x^2)$. Hence for $0 \leqslant x \leqslant \frac{1}{2}$,

$$f' \leqslant (1-P) k^2 x, \quad f \leqslant \tfrac{1}{2}(1-P) k^2 x^2. \tag{26}$$

Substituting back into (21) gives

$$f' \geqslant \tfrac{3}{4} c x - \tfrac{1}{4}(1-P)^2 k^4 x^4 \geqslant \tfrac{1}{2} c x \tag{27}$$

for $0 \leqslant x \leqslant \beta\delta$, where $\beta = [Q/(1-P)]^{\frac{2}{3}} < 1$. Thus from (13) and (26)

$$0 \leqslant \frac{F}{\sin \alpha} \leqslant \tfrac{1}{2}(1-P) k x^2 \quad (0 \leqslant x \leqslant \tfrac{1}{2}).$$

Also, since $F' = 2\nu(1-x^2) f' - 4\nu x f$, we have by (26) and (27)

$$\tfrac{1}{2} Q k x \{1 - (1 + 2[(1-P)/Q]) x^2\} \leqslant F' \leqslant (1-P) k x,$$

valid for $0 \leqslant x \leqslant \beta\delta$ (and of course $0 \leqslant x \leqslant \tfrac{1}{2}$, as always). In particular, for k sufficiently large

$$|F/\sin \alpha| \leqslant M^2 Q^{\frac{1}{3}} k^{-\frac{1}{3}} \quad (x = M\delta, M < 1/2\delta)$$

and

$$\frac{M}{3} Q^{\frac{2}{3}} k^{\frac{1}{3}} \leqslant F' \leqslant \frac{4M}{3} Q^{\frac{2}{3}} k^{\frac{1}{3}} \quad (x = M\delta, M < 1).$$

Recalling the relations (2), *this proves the assertions concerning the asymptotic behaviour of the velocity components v_α and v_R in the boundary layer.*

It is a remarkable fact that, even though the fluid is slowed near the wall by the adherence condition and the pressure at the wall increases outward (when $P > 0$), nevertheless there is a strong *outward* component of velocity in the boundary layer. The explanation of this phenomenon is that the pressure gradient is for the most part used to oppose the centrifugal force of the vortex flow, leaving a net outward driving force on the radial velocity.

6. *Appendix*

Here we shall give the proof of lemma 6. By lemmas 4 and 5, f is either ultimately positive or everywhere negative. In the former case we must have $P < 1$, and the required result then follows from the displayed inequality of lemma 4. We may consequently suppose from there on that f is everywhere negative.

Consider the function $\omega(x) = -1/2(1-x)$. Since $G(x) = O(1-x)$ as $x \to 1$ (see lemma 2) it is easily checked that

$$\omega' < k^2 \frac{G(x)}{(1-x^2)^2} - \omega^2$$

for x sufficiently near 1, say $x_0 \leqslant x < 1$. Consequently if $f = \omega$ at some point $x_1 \geqslant x_0$, we have $f > \omega$ thereafter. It follows that either

$$f > \omega \quad \text{or} \quad f < \omega$$

for all x sufficiently near 1. We shall show that only the first case can actually occur.

Thus suppose for contradiction that $f < \omega$ on some interval $x_2 \leqslant x < 1$. Then by (17), for $x \geqslant x_2$,

$$\Omega' = \text{const.} \exp\left(-2\int_{x_2}^{x} f \, dx\right) \geqslant \text{const.} \exp\left(-2\int_{x_2}^{x} \omega \, dx\right) = \text{const.} \frac{1-x_2}{1-x}.$$

Since the function on the right-hand side is not integrable, this contradicts the fact that $\Omega \to 1$ as $x \to 1$.

Having shown that $f > -1/2(1-x)$ for x near 1, let us now make the change of variables

$$\phi = -\sqrt{(1-x)}f$$

in (14). Recalling that $f < 0$, this gives the result

$$\phi' = \{1 + 2(1-x)f\}\frac{f}{2\sqrt{(1-x)}} - k^2\sqrt{(1-x)}\frac{G(x)}{(1-x^2)^2} \leqslant \frac{\text{const.}}{\sqrt{(1-x)}}$$

for all x near 1. By integration $\phi \leqslant$ constant, and therefore $f \geqslant -\kappa/\sqrt{(1-x)}$ for some positive constant κ and for all x near 1. Substituting this in (14) we obtain

$$f' \geqslant k^2\frac{G(x)}{(1-x^2)^2} - \kappa^2\frac{1}{(1-x)} \geqslant -\frac{\text{const.}}{1-x}.$$

Hence by integration $f \geqslant -\text{const.}\ln 1/(1-x)$. Since f was assumed to be negative, this completes the proof of lemma 6.

Chapter III. General theory

We show in this chapter that the basic set of equations (14), (15) has a solution for certain combinations of the parameter values k and P, but not for others.

7. *Non-existence of solutions for certain parameter values*

It is a surprising and somewhat paradoxical fact that the problem as posed may not have solutions for high Reynolds numbers and certain fixed values of P. This was already pointed out by Goldstik for the special case $P = 1$. Here we shall investigate this phenomenon in more detail (see also the remarks in chapter IV).

Consider first the case $P \geqslant 1$. Then f is negative and Ω convex, according to theorem 2. Consequently $\Omega(x) \leqslant x$ and from (12) we find after some calculation

$$G(x) \leqslant (1+x)^2 \ln(1+x) + (1-x)^2 \ln(1-x) - 4x \ln 2 + (3-P)(x-x^2) \equiv I(x).$$

By numerical computation† we obtain

$$I(x) \leqslant -\tfrac{1}{2}x(1-x^2)^2 + (1-P)(x-x^2) \leqslant -(\tfrac{1}{2}P)x(1-x^2)^2.$$

Consequently if a solution exists, then

$$f' \leqslant -\tfrac{1}{2}Pk^2x - f^2.$$

It is apparent that the solution f cannot exist on the entire interval from 0 to 1 when the coefficient of x is large. More specifically, consider the comparison equation

$$\sigma' = -d^2x - \sigma^2, \quad \sigma(0) = 0.$$

By explicit integration, Goldstik (p. 921) found that σ cannot be continued over the interval $[0, 1)$ whenever $d > 3\mu_1/2$, where μ_1 is the first root of the (Bessel) equation $J_{-\frac{1}{3}}(\mu) = 0$. The same conclusion therefore holds for the function f by comparison. Consequently we have proved

PROPOSITION 1. *There can be no solution of* (14), (15) *if* $P \geqslant 1$ *and* $Pk^2 > 9\mu_1^2/2 \approx 16$.

In view of theorem 2 and the result just proved, negative solutions f can exist only when k is less than 4. Therefore the most interesting class of vortex motions of the type (2) will have either f positive or else f first negative and then afterwards positive (such flows correspond to a downward swirling motion in the vortex, as pointed out in §4).

Consider now solutions f which are first negative and then positive thereafter. Let the zero of f occur at the position $x = a$, $0 < a < 1$. Then the following result holds.

PROPOSITION 2. *For fixed* a, $0 < a < 1$, *we must have*

$$P \leqslant \frac{1}{a}\left\{1 + \frac{1-a}{a^2}\ln(1-a^2)\right\}$$

in order for a solution to exist with $f(a) = 0$.

---

† See Goldstik, p. 920. I take it to be legitimate to check an inequality between two elementary functions by a numerical (digital) calculation of the functions involved. In any case, an analytic proof is available for the inequality $I(x) \leqslant -\frac{2}{5}Px(1-x^2)^2$, which would lead to essentially the same conclusions.

To prove this, note that by theorem 2 the function Ω is convex for $0 \leqslant x < a$ and concave for $a < x < 1$. Consequently, taking into account the boundary conditions on Ω, it is easy to see that

$$\Omega(x) \leqslant \begin{cases} x/a, & 0 \leqslant x \leqslant a, \\ 1, & a \leqslant x \leqslant 1. \end{cases}$$

Hence by direct evaluation of the integrals involved in the definition of G, we find that

$$G(a) \leqslant (1-a) \left\{ 1 - aP + \frac{1-a}{a^2} \ln (1-a^2) \right\}. \tag{28}$$

On the other hand, $G(a) > 0$ as noted in the proof of lemma 4. Comparing this inequality with (28) yields the required result.

Proposition 2 implies that motions in which f (and consequently F) have both positive and negative values can exist only at certain values of the parameters. It should be noted in particular that the function

$$\frac{1}{a} \left\{ 1 + \frac{1-a}{a^2} \ln (1-a^2) \right\}$$

is less than 1 for $0 < a < 1$, and tends to 1 as $a \to 0, 1$. We consider finally the case when f is everywhere positive.

PROPOSITION 3. *In order for a positive solution f to exist we must have*

$$P \leqslant K(k),$$

where $K(k)$ is a positive monotone function of k, which increases from $3 - 4 \ln 2 (\approx 0.224)$ to 1 as k goes from 0 to ∞.

Proof. By integration of (17) from 0 to 1 it is easy to obtain the relation

$$\Omega'(0) = \left\{ \int_0^1 \exp \left(-2 \int_0^x f \, dx \right) dx \right\}^{-1}.$$

Recalling that $P < 1$ in the present case, we see by lemma 4 that the right-hand side is in turn

$$\leqslant \left\{ \int_0^1 \exp \left(\tfrac{1}{2}(P-1) \, k^2 \int_0^1 \ln \frac{1}{1-x} \, dx \right) dx \right\}^{-1} = \exp \left(\tfrac{1}{2}(1-P) \, k^2 \right) \equiv A.$$

Therefore since Ω is concave (cf. theorem 2) it follows that

$$\Omega(x) \leqslant \begin{cases} Ax, & 0 \leqslant x \leqslant 1/A, \\ 1, & 1/A \leqslant x \leqslant 1. \end{cases}$$

Thus by lemma 2 and the fact that $G'(0) \geqslant 0$ for a positive solution f, we find

$$P = 2 \int_0^1 \frac{\Omega^2 \, dt}{(1+t)^2} - G'(0) \leqslant 4A - 4A^2 \ln \frac{1+A}{A} - 1 \equiv \chi(A). \tag{29}$$

Now
$$\frac{d\chi}{dA} = 4A \left\{ \frac{1+2A}{A(1+A)} - 2 \ln \frac{1+A}{A} \right\} \quad (A > 1).$$

The quantity in braces is easily found to be a decreasing function, and has the value zero when $A = \infty$. It follows therefore that $d\chi/dA > 0$ and that χ is an increasing function.

If we fix k, the function $\chi(A)$ therefore decreases from 1 to $3 - 4 \ln 2$ as P increases from $-\infty$ to 1 (and A correspondingly decreases from ∞ to 1). Hence for each k inequality (29) places an upper

bound $K(k)$ on P. It is easy to check that $K(k)$ is increasing, and varies from $3 - 4\ln 2$ when $k = 0$ to 1 when $k = \infty$.

Proposition 3 shows that for k small, P can be little larger than $3 - 4\ln 2$ in order for a positive solution to exist, while conversely for large k only values of P approaching 1 are ruled out. We shall see later (corollary 4) that the qualitative behaviour indicated by this proposition is essentially best possible, since positive solutions are there proved to exist for all P not exceeding $3 - 4\ln 2$, and for P arbitrarily near 1 if k is suitably large.

8. *Existence of solutions* (*I*)

In this section we shall establish the existence of solutions for certain ranges of the parameters k and P. Note that according to the work of the preceding section one cannot expect to solve the given problem for arbitrary values of these parameters.

Let $\tilde{k}$, $\tilde{P}$ be fixed parameter values, and let $\tilde{f}$, $\tilde{\Omega}$ be a pair of functions satisfying the conditions $\tilde{f}(0) = \tilde{\Omega}(0) = 0$ and such that also

$$\tilde{f} = O\left(\ln\frac{1}{1-x}\right) \quad \text{and} \quad 0 \leqslant \tilde{\Omega} \leqslant 1. \tag{30}$$

Then $\tilde{f}$, $\tilde{\Omega}$ will be called a *subsolution* of the system (14), (15) provided

$$\left.\begin{array}{l} \tilde{f}' + \tilde{f}^2 \leqslant \tilde{k}^2 \dfrac{\tilde{G}(x)}{(1-x^2)^2} \\[2mm] \tilde{\Omega}'' + 2\tilde{f}\tilde{\Omega}' \geqslant 0 \end{array}\right\} \quad (0 \leqslant x < 1), \tag{31}$$

where $\tilde{G}$ denotes the function (12) with Ω replaced by $\tilde{\Omega}$ and P by $\tilde{P}$. The following main result now holds.

THEOREM 3. *Let* $\tilde{f}$, $\tilde{\Omega}$ *be a subsolution for given parameter values* $\tilde{k}$ *and* $\tilde{P}$. *Then there exists a solution* f, Ω *of the system* (14), (15) *for the same* k *and all values of* P *not exceeding* $\tilde{P}$. *Moreover*

$$f \geqslant \tilde{f}, \quad \Omega \geqslant \tilde{\Omega}.$$

Proof. Consider the method of successive approximations given by the scheme

$$\Omega_0 \equiv 1$$

and for $n \geqslant 1$, $\qquad G_n(x) = 2(1-x)^2 \displaystyle\int_0^x \frac{t\Omega_{n-1}^2}{(1-t^2)^2}\,\mathrm{d}t + 2x \int_x^1 \frac{\Omega_{n-1}^2\,\mathrm{d}t}{(1+t)^2} - P(x - x^2),$

$$f_n' + f_n^2 = k^2 \frac{G_n(x)}{(1-x^2)^2}, \quad f_n(0) = 0,$$

$$\Omega_n'' + 2f_n\Omega_n' = 0, \quad \Omega_n(0) = 0, \quad \Omega_n(1) = 1.$$

We shall show that this scheme is well defined, and that for $0 \leqslant x < 1$

$$f_1 \geqslant f_2 \geqslant \ldots \geqslant \tilde{f}, \quad \Omega_0 \geqslant \Omega_1 \geqslant \Omega_2 \geqslant \ldots \geqslant \tilde{\Omega}. \tag{32}$$

The proof will be by induction. We establish first that f_1 and Ω_1 are well defined, and that

$$f_1 \geqslant \tilde{f}, \quad \Omega_0 \geqslant \Omega_1 \geqslant \tilde{\Omega}. \tag{33}$$

To begin with, because $0 \leqslant \tilde{\Omega} \leqslant 1$ and $\tilde{P} \geqslant P$ it follows that

$$\tilde{G}(x) \leqslant G_1(x) \equiv (1-P)(x - x^2).$$

Since
$$f_1' = k^2 \frac{G_1(x)}{(1-x^2)^2} - f_1^2 \tag{34}$$

the function $\tilde{f}$ therefore serves as a minorant for f_1, while

$$\max\left(0, \tfrac{1}{4}(1-P)\,k^2\ln\frac{1}{1-x}\right)$$

serves as a majorant (see lemma 4). Thus according to standard results in the theory of ordinary differential equations, (34) has a solution f_1 on the full interval $0 \leqslant x < 1$, and the first inequality of (33) holds.†

Using (30) we see from the above construction that $f_1 = O(\ln 1/(1-x))$. Hence f_1 is integrable and Ω_1 can be determined by quadrature as noted in the proof of lemma 1. It remains to prove the second inequality of (33). We require the following comparison lemma.

Suppose that for $0 \leqslant x < 1$ we have

$$\Omega'' + 2f\Omega' = 0, \quad \bar{\Omega}'' + 2\bar{f}\bar{\Omega}' \geqslant 0$$

together with the boundary conditions $\quad \Omega(0) = \bar{\Omega}(0) = 0$

and $\quad \Omega(1) = 1, \quad \bar{\Omega} \leqslant 1 \quad$ *for x near 1.*

Then if $f \geqslant \bar{f}$ we have $\Omega \geqslant \bar{\Omega}$.

Proof. Obviously $\quad (\bar{\Omega} - \Omega)'' + 2\bar{f}(\bar{\Omega} - \Omega)' \geqslant 2(f - \bar{f})\,\Omega' \geqslant 0,$

since Ω' is certainly positive (see lemma 1). Multiplying through by the integrating factor $\exp\left(2\int_0^x \bar{f}\,\mathrm{d}x\right)$ we find

$$\left\{(\bar{\Omega} - \Omega)'\exp\left(2\int_0^x \bar{f}\,\mathrm{d}x\right)\right\}' \geqslant 0. \tag{35}$$

Now suppose for contradiction that the function $\bar{\Omega} - \Omega$ takes on positive values. Then it has a positive maximum at some point $x_0 < 1$. Integrating (35) from x_0 to $x > x_0$, we get

$$(\bar{\Omega} - \Omega)' \geqslant 0, \quad \bar{\Omega} - \Omega \geqslant \bar{\Omega}(x_0) - \Omega(x_0) > 0 \quad \text{for} \quad x \geqslant x_0,$$

contradicting the boundary conditions at $x = 1$. The lemma is therefore proved.

Applying the lemma with $\Omega = \Omega_1$, $\bar{\Omega} = \tilde{\Omega}$ now establishes $\Omega_1 \geqslant \tilde{\Omega}$. The remaining inequality of (33), namely $\Omega_0 \geqslant \Omega_1$, is obvious since $0 \leqslant \Omega_1 \leqslant 1$ (see lemma 1).

Now consider the induction hypothesis

$$f_1 \geqslant \ldots \geqslant f_n \geqslant \tilde{f}, \quad \Omega_0 \geqslant \Omega_1 \geqslant \ldots \geqslant \Omega_n \geqslant \tilde{\Omega}, \tag{36}$$

which by (33) surely holds when $n = 1$. Supposing (36) to hold for $n(\geqslant 1)$ we shall verify it for $n+1$. From the fact that $P \leqslant \tilde{P}$ we have easily

$$G_n(x) \geqslant G_{n+1}(x) \geqslant \tilde{G}(x).$$

Using G_n and $\tilde{G}$ as comparison functions, we see that f_n serves as a majorant for f_{n+1} and $\tilde{f}$ serves as a minorant. Thus f_{n+1} is well defined and

$$f_n \geqslant f_{n+1} \geqslant \tilde{f},$$

i.e. the first inequality of (36) holds for $n+1$. Moreover, since f_{n+1} is $O(\ln 1/(1-x))$ it is clear that Ω_{n+1} is well defined (see the earlier discussion of Ω_1).

By applying the lemma, first with $\Omega = \Omega_n$, $\bar{\Omega} = \Omega_{n+1}$ and then with $\Omega = \Omega_{n+1}$, $\bar{\Omega} = \tilde{\Omega}$, we obtain $\Omega_n \geqslant \Omega_{n+1} \geqslant \tilde{\Omega}$. Hence the second inequality of (36) also holds for $n+1$.

† See Coddington & Levinson (1955) chapter 1.

The validity of (36) is thereby established for all n, and accordingly (32) is proved. That is, the sequences $\{\Omega_n\}$, $\{f_n\}$ are monotonically decreasing and bounded below for fixed values of x. Consequently they converge to limits:

$$\Omega_n \to \Omega, \quad f_n \to f.$$

Evidently $\Omega \geqslant \tilde{\Omega}$ and $f \geqslant \tilde{f}$. It remains to show that these functions are solutions of (14), (15).

In view of (36) the functions f_n' are uniformly bounded on any closed subinterval of $[0, 1)$. Thus f_n converges uniformly on closed subintervals of $[0, 1)$. Moreover the functions f_n are uniformly integrable according to the construction.

It now follows from the explicit integral formula for Ω_n in terms of f_n that Ω_n converges uniformly to Ω on $0 \leqslant x \leqslant 1$. Furthermore, Ω is twice differentiable on $0 \leqslant x < 1$ and

$$\Omega'' + 2f\Omega' = 0.$$

Having shown that Ω_n converges uniformly to Ω, it follows that G_n converges uniformly to G; hence letting n tend to infinity in the relation

$$f_n = \int_0^x \left(k^2 \frac{G_n(x)}{(1-x^2)^2} - f_n^2 \right) dx$$

we find that f is differentiable on $0 \leqslant x < 1$ and is a solution of (14). This completes the proof of theorem 3.

COROLLARY 1. *If* (14), (15) *is solvable for parameter values* $\tilde{k}$, $\tilde{P}$, *it is solvable for the same k and all* $P \leqslant \tilde{P}$. *For the solutions in question, if P decreases then f increases.*

If the solution $\tilde{f}$ is positive, then the problem is solvable with f positive for all $k \geqslant \tilde{k}$ and $P \leqslant \tilde{P}$. For the solutions in question, if P decreases and k increases then f increases. Finally, if $\tilde{P} \geqslant 1$ then the problem is solvable for all $k \leqslant \tilde{k}$ and $P \leqslant \tilde{P}$.

Proof. The first statement follows at once from theorem 3, since any solution of (14), (15) is also a subsolution (see lemma 6). Now suppose $\tilde{f}$ is positive. The main proof then applies word for word (with $k \geqslant \tilde{k}$), once we notice that $k^2\tilde{G} \geqslant \tilde{k}^2\tilde{G}$ because $\tilde{G}$ is positive.

When $\tilde{P} \geqslant 1$ we note that $\tilde{G}$ is negative (lemma 3). Thus $k^2\tilde{G} \geqslant \tilde{k}^2\tilde{G}$ when $k \leqslant \tilde{k}$. The main proof then applies word for word.

In the statement of corollary 1, the phrase ' (and indeed by successive approximations)' could be added following the words 'it is solvable'. In particular, this means that if a solution (positive solution) exists at all for given k, P, then the method of successive approximations will converge to a solution (positive solution) for these parameter values. We shall take advantage of this remark in §10.

COROLLARY 2. *The system* (14), (15) *is solvable with f positive provided that*

$$P \leqslant 3 - 4\ln 2 \approx 0.224. \tag{37}$$

Proof. We shall show that $\tilde{f} \equiv 0$, $\tilde{\Omega} \equiv x$ is a subsolution. Indeed, it is enough to verify that the function $\tilde{G}$ associated with $\tilde{\Omega} = x$ is positive. By lemma 2, however,

$$\tilde{G}'(0) = 2\int_0^1 \frac{t^2\,dt}{(1+t)^2} - P$$

$$= 3 - 4\ln 2 - P \geqslant 0,$$

and consequently $\tilde{G} > 0$ by lemma 3. This completes the proof.

It should be noted that *condition* (37) *is best possible*. Indeed, if $P > 3 - 4\ln 2$ then for all sufficiently small k no positive solution can exist, according to the remark following proposition 3 in §7.

COROLLARY 3. *There exists an absolute constant* $\lambda \approx 2.85$ *such that* (14), (15) *is solvable provided that*

$$Pk^2 < \lambda^2,$$

but is not solvable when $(P-1)\, k^2 > \lambda^2$.

Proof. We seek a subsolution with $\tilde{\Omega} \equiv 0$. Then $\tilde{G} \equiv P(x^2 - x)$, and we can choose $\tilde{f}$ as the solution of the equation

$$\tilde{f}' + \tilde{f}^2 = -Pk^2 \frac{x}{(1-x)\,(1+x)^2} \quad (0 \leqslant x < 1),$$

with corresponding boundary conditions

$$\tilde{f}(0) = 0, \quad \tilde{f} = O(\ln(1/1-x)).$$

It is evident† that there is a constant $\lambda > 0$ such that this problem is solvable if and only if $Pk^2 < \lambda^2$. The approximate value 2.85 for λ was determined by a simple computer program. This proves the first part of the corollary.

Now for any solution of (14), (15), lemma 3 implies

$$f' + f^2 \leqslant (1-P)\, k^2 \frac{x}{(1-x)\,(1+x)^2}.$$

From the remarks above we have $-(1-P)\, k^2 < \lambda^2$, and the remaining part of the corollary follows at once (note that this part of the result overlaps proposition 1 in §7).

The preceding three corollaries contain the main existence theorem of this section. In the next section we shall use a different method to obtain further results.

9. *Existence of solutions* (II)

The results of the previous section establish the existence of two of the three possible types of solutions described by theorem 2. Namely when $P \leqslant 0.224$ we obtain a positive solution f, while when $P \geqslant 1$ and $Pk^2 < 8.2$ we find a negative decreasing solution. These results do not, however, shed light one way or the other on the existence of the remaining type of solution, namely one which is first negative, has a single zero, and is positive thereafter. This type of solution seems to be

† To see this, let S be the set of values of Pk^2 for which the problem is solvable. Clearly S contains zero, and if μ is in S then any value $\mu' < \mu$ is also in S. Consequently it is only necessary to show that S is open on the right. This being the case, let μ be in S and let g be the corresponding solution. Put

$$\bar{g} = g - \ln \frac{1}{1-x}.$$

Then an easy calculation shows that

$$\bar{g}' \leqslant -(\mu+1) \frac{x}{(1-x)\,(1+x)^2} - \bar{g}^2$$

on some interval $1 - \gamma \leqslant x < 1$. Now let $\epsilon(<1)$ be a small positive number such that the equation

$$h' = -(\mu+\epsilon) \frac{x}{(1-x)\,(1+x)^2} - h^2$$

has a solution on the interval $0 \leqslant x \leqslant 1 - \gamma$, with $h(0) = 0$, $h(1-\gamma) \geqslant \bar{g}(1-\gamma)$ (the existence of such an ϵ follows from the fact that we can choose $h = g$ when $\epsilon = 0$, together with standard perturbation theory; Coddington & Levinson, p. 29). By the previous construction h can be continued throughout the remaining interval $1 - \gamma \leqslant x < 1$, with $\bar{g} \leqslant h \leqslant g$. Hence $\mu + \epsilon$ is in S, as required.

of particular interest; our purpose here will be to show that such solutions do in fact exist, and indeed can exist at arbitrarily high values of k.

As motivation for our approach, we observe that for such solutions we have necessarily (see lemma 2 and theorem 2)

$$G'(0) = 2\int_0^1 \frac{\Omega^2 \, dt}{(1+t)^2} - P < 0.$$

Let us therefore consider $Q = G'(0)$ as the basic parameter, and correspondingly put

$$P = 2\int_0^1 \frac{\Omega^2}{(1+t)^2} \, dt - Q. \tag{38}$$

Our object is now to solve (14), (15), (38) *for fixed values of Q.*

Eliminating P from (12) by using (38), we can write $G(x)$ in the form

$$G(x) = -2\int_0^x \frac{(x-t)(1-tx)}{(1-t^2)^2} \Omega^2 \, dt + 2x^2 \int_0^1 \frac{\Omega^2 \, dt}{(1+t)^2} + Q(x - x^2). \tag{39}$$

It is apparent from (39) that the method of theorem 3 will not work in the present case, since G is no longer a monotone functional of Ω; that is, when Q remains fixed we no longer have $\bar{G} \geqslant G$ when $\bar{\Omega} \geqslant \Omega$. Nevertheless, a modified approach making use of well-known fixed point theorems can be given.

Consider functions f which are continuous on $0 \leqslant x < 1$ and satisfy the condition

$$f = O(\ln 1/(1-x)). \tag{40}$$

Correspondingly, let us determine Ω by the equation

$$\Omega'' + 2f\Omega' = 0 \tag{41}$$

together with the usual boundary conditions; finally let g satisfy the differential equation

$$g' + g^2 = k^2 \frac{G(x)}{(1-x^2)^2}, \quad g(0) = 0,$$

where $G(x)$ is the function introduced above. This defines a transformation

$$f \to g = Tf.$$

Note that g need not be determined on the entire interval $0 \leqslant x < 1$; that is, it may diverge to $-\infty$ at some point $x = b < 1$. In any case, our purpose is to find a fixed point f of this transformation, that is, a function f such that Tf exists for $0 \leqslant x < 1$ and satisfies $f = Tf$.

To this end, we shall employ a form of the Schauder fixed point theorem. Let X be the space of continuous functions on the interval $0 \leqslant x < 1$ with the metric

$$\text{dist}\,(f,g) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{M_m(f,g)}{1+M_m(f,g)}, \quad f, g \in X,$$

where

$$M_m(f,g) = \sup_{0 \leqslant x \leqslant 1 - \frac{1}{m}} |f(x) - g(x)|.$$

Clearly a sequence of functions f_n converges in X to a function f if and only if $f_n \to f$ uniformly on any compact subinterval of $[0, 1)$. Moreover, under the topology induced by the metric, X is obviously a locally convex linear topological space (see Dunford & Schwartz 1958, for definitions).

This being the case, if $\hat{X}$ is a closed convex subset of X which is continuously mapped by T into a conditionally compact subset of itself, then T has a fixed point in $\hat{X}$ (Dunford & Schwartz, p. 456; the result stated there is slightly different, but is easily seen to imply the present form). It should be noted that since X is a metric space, a set A is conditionally compact in X if every sequence of points in A has a subsequence converging to a point in X.

THEOREM 4. *The system* (14), (15) *is solvable, with $G(x)$ given by* (39), *provided that*

$$Qk^2 > -\lambda^2,$$

where λ is the constant of corollary 3.

Proof. It is necessary to determine an appropriate set $\hat{X}$ for the application of Schauder's theorem.

Let f be a continuous function on $0 \leqslant x < 1$ satisfying condition (40). We examine in detail the transformation $g = Tf$. In particular, with Ω defined by (41), it is clear that

$$G(x) \geqslant Q(x - x^2)$$

according to lemma 3. Consequently we have

$$g' + g^2 \geqslant Qk^2 \frac{x}{(1-x)(1+x)^2} \quad (g(0) = 0).$$

Let the function τ be defined by the equation

$$\tau' + \tau^2 = Qk^2 \frac{x}{(1-x)(1+x)^2}.$$

As already remarked in the proof of corollary 3 this equation is solvable for the boundary conditions

$$\tau(0) = 0, \quad \tau = O(\ln 1/(1-x)),$$

provided that $Qk^2 > -\lambda^2$. Hence under the hypothesis of the theorem, comparison of the functions g and τ yields the relation

$$g(x) \geqslant \tau(x).$$

Also by lemma 3, one has

$$G(x) \leqslant (1-P)(x - x^2) \leqslant (1+Q)(x - x^2)$$

since $P > -Q$ by (38). Consequently, as in the proof of lemmas 4 and 5,

$$g(x) \leqslant \begin{cases} 0, & Q \leqslant -1, \\ \frac{1}{4}(1+Q)k^2 \ln 1/(1-x), & Q > -1. \end{cases}$$

Combining the preceding estimates, it is clear that under the conditions of the theorem T is well-defined over the full interval $0 \leqslant x < 1$.

We now choose $\hat{X}$ to be the set of all functions $f \in X$ such that

$$\tau(x) \leqslant f(x) \leqslant 0 \qquad \text{if} \quad Q \leqslant -1$$

$$\tau(x) \leqslant f(x) \leqslant \tfrac{1}{4}(1+Q)k^2 \ln \frac{1}{1-x} \quad \text{if} \quad Q > -1.$$

Clearly Tf is well defined for all $f \in \hat{X}$, and moreover we have $T\hat{X} \subset \hat{X}$ according to the first part of the proof. Also using the fact that $g = Tf$ satisfies

$$g' = k^2 \frac{G(x)}{(1-x^2)^2} - g^2,$$

we see easily that $T\hat{X}$ consists of functions which are uniformly continuous on closed subintervals of $[0, 1)$. Hence by Arzela's theorem $T\hat{X}$ is conditionally compact in X.

The set $\hat{X}$ is obviously closed and convex in X. Finally, the mapping T is continuous on $\hat{X}$. Indeed, using the explicit integral for Ω in terms of f, one sees that if f converges to f_0 in $\hat{X}$ then Ω converges uniformly to Ω_0 on $0 \leqslant x \leqslant 1$. Consequently G converges to G_0 uniformly on closed subintervals of $0 \leqslant x < 1$, and g in turn converges uniformly to g_0 on such subintervals.

The conditions of Schauder's theorem are therefore satisfied, and accordingly there exists a fixed point f of T. It remains only to note that when Ω is defined by (41) and the usual boundary conditions, then f, Ω satisfy (14), (15). This completes the proof of the theorem.

The preceding theorem obviously establishes the existence of solutions of (14), (15) for all non-negative values of $Q = G'(0)$. These solutions correspond to parameter values $P < 1$, and are of course such that f is positive.

We can in fact be more explicit. Consider a fixed value $k \geqslant e$ and let $Q = k^{-\frac{1}{2}}$. For the corresponding solution we have $P > 1 - 12k^{-\frac{1}{2}}(1 + \ln k)$ by (25). Hence applying corollary 1, there exist positive solutions whenever

$$P \leqslant 1 - 12k^{-\frac{1}{2}}(1 + \ln k), \quad (k \geqslant e)$$

Combining this with the result of corollary 2 now proves

COROLLARY 4. *There exist positive solutions of* (14), (15) *whenever*

$$P \leqslant \max\left(3 - 4\ln 2, \; 1 - 12k^{-\frac{1}{2}}(1 + |\ln k|)\right).$$

The following result answers the question raised at the beginning of the section.

COROLLARY 5. *For each $k > 0$ there exist solutions of* (14), (15) *such that f is first negative, has a single zero on $0 < x < 1$, and is positive thereafter.*

Proof. By the main theorem of the section, there exists for each fixed $k > 0$ a solution with $Q = -\lambda^2/2k^2$. If the corresponding parameter value $P(k)$ is less than 1, it is clear by theorem 2 that the solution in question is of the required sort and there is nothing more to prove. We may thus suppose that $P(k) \geqslant 1$.

Then by corollary 1 the system (14), (15) is solvable for the fixed value k and for all $P < 1$. In particular, a solution exists when $P = \frac{1}{2}(1 + K(k))$ where the function $K(k)$ is defined in proposition 3. This solution cannot be positive (by proposition 3), and is not negative (since $P < 1$). The only remaining possibility is that the solution has the required behaviour. This completes the proof.

Discussion of results of chapter III

It is convenient to associate a point in the (P, k^{-2}) plane with each pair of parameter values P and k. Then by corollary 1 (§ 8) positive solutions of the system (14), (15) exist for all parameter values which lie on the left-hand side of a certain fixed monotonically decreasing curve Σ in the (P, k^{-2}) plane. According to corollary 4 and proposition 3 this curve must satisfy the condition

$$\max\left(3 - 4\ln 2, \; 1 - 12k^{-\frac{1}{2}}(1 + |\ln k|)\right) \leqslant P \leqslant K(k). \tag{42}$$

Consequently, taking into account proposition 3, the curve approaches the asymptotic value $P = 3 - 4\ln 2$ as k^{-2} tends to infinity, and tends to 1 as k^{-2} tends to zero.

Similarly, by corollary 1 the zone in which solutions of any kind can exist must be bounded on the right by a second curve Σ' which is monotonically increasing when $P \geqslant 1$, and has a single-valued projection on the k^{-2} axis when $P < 1$. This curve obviously lies to the right of the first one. Moreover by corollary 3 and proposition 1 it must satisfy the relation

$$\lambda^2 k^{-2} \leqslant P \leqslant \max\left(1, \tfrac{9}{2}\mu_1^2 k^{-2}\right). \tag{43}$$

These qualitative results are illustrated in figure 1. Specifically, the region A in which positive solutions exist is shown bounded by a monotonically decreasing curve with the properties noted
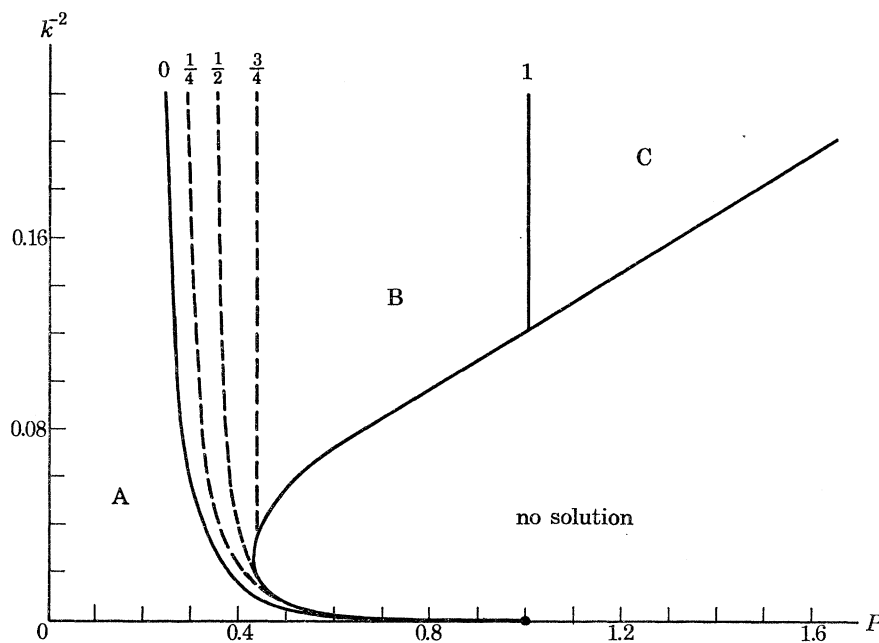
FIGURE 1. (P, k^{-2}) parameter diagram.

above; similarly, an appropriate curve satisfying (43) is indicated as the boundary of the domain in which solutions of any sort can exist. Zones B and C are separated by the line $P = 1$. In zone B the solution f is first negative and afterwards positive, while in zone C the solution is everywhere negative (see theorem 2).

Determination of the precise form of the curves Σ and Σ' apparently requires the numerical calculation of solutions. In the following section we describe the results of such a calculation and present some representative solutions for several typical values of the parameter k. The curves in figure 1 are in fact drawn so as to be in agreement with this work (as well as being consistent with the preceding theory). The dotted lines in zone B are curves along which $a = \text{constant}$ (where $F(a) = 0$), as interpolated from the numerical results. These curves agree in all respects with the conditions of proposition 2 and corollary 5. Finally, the line $P = 1$ separating zones B and C is found to terminate at the value $k^{-2} \approx 0.122$, that is, when $k \approx 2.86$. Consequently, solutions involving a central updraft can exist only at values of k not greater than this.
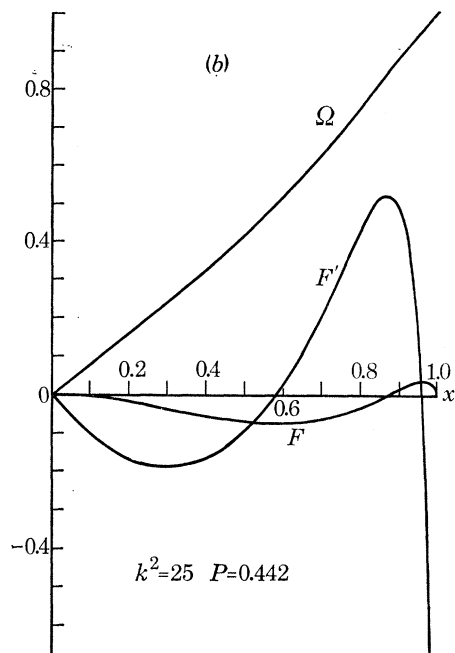
FIGURE 2 b. Numerical solution when $k = 5$, $P = 0.442$ (zone B).
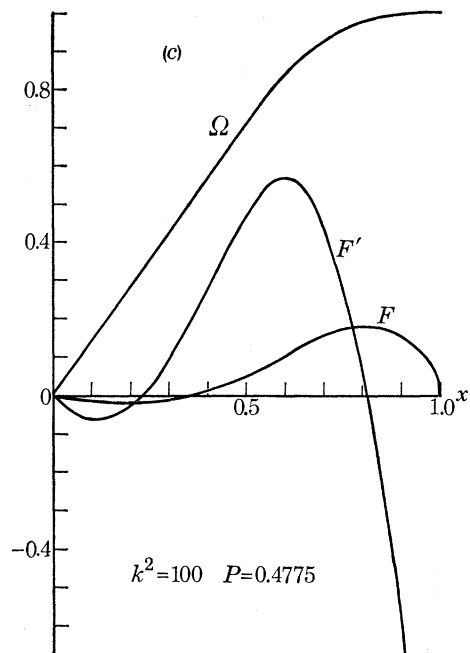
FIGURE 2 c. Numerical solution when $k = 10$, $P = 0.4775$ (zone B).
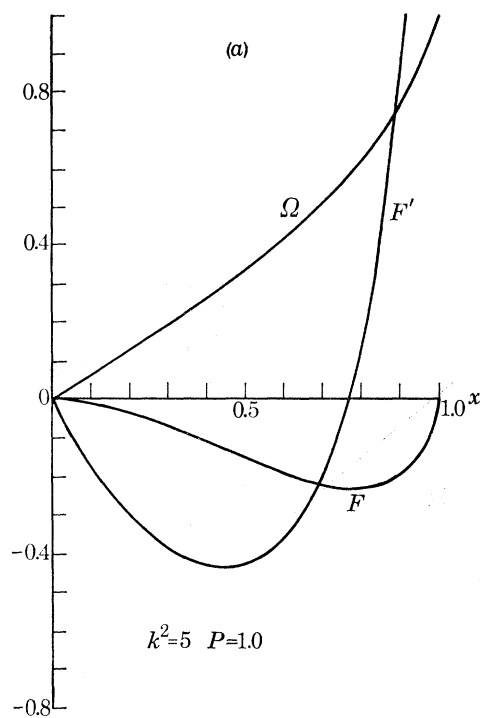
FIGURE 2 a. Numerical solution when $k = 2.236$, $P = 1$ (zone C).
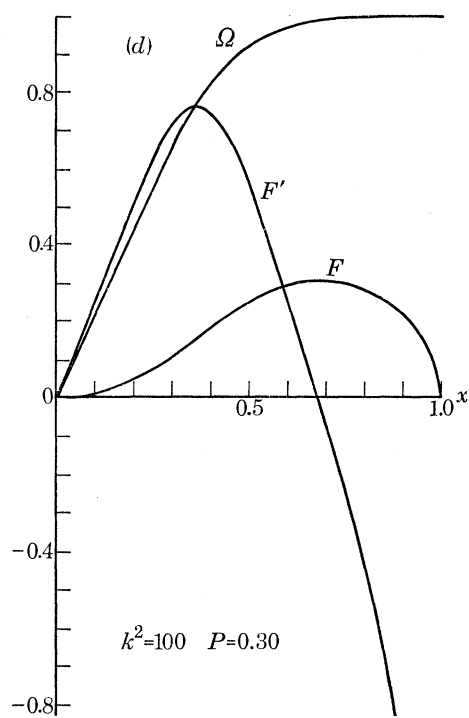
FIGURE 2 d. Numerical solution when $k = 10$, $P = 0.3$ (zone A).
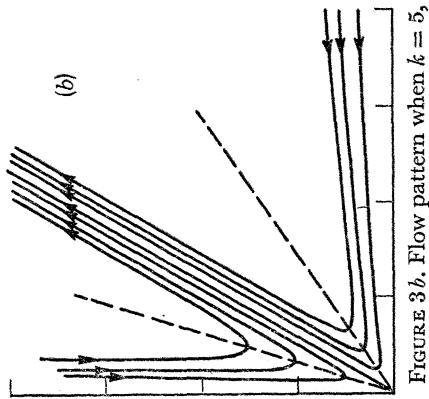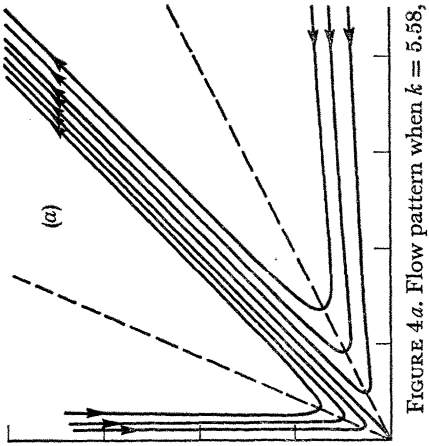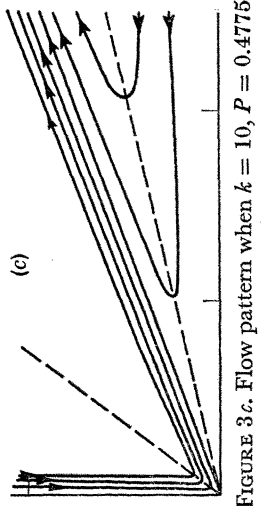
J. SERRIN

FIGURE 3c. Flow pattern when $k = 10$, $P = 0.4775$ (zone B).

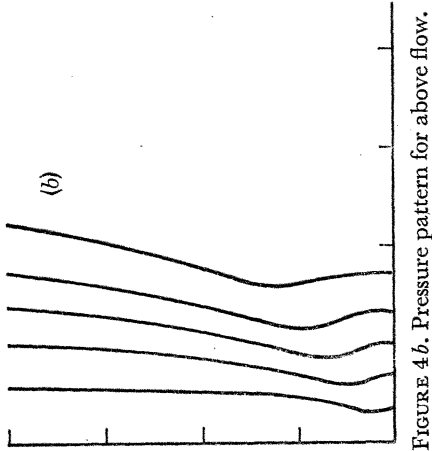FIGURE 3d. Flow pattern when $k = 10$, $P = 0.3$ (zone A).

FIGURE 4a. Flow pattern when $k = 5.58$, $P = 0.428$ (zone B).

FIGURE 4b. Pressure pattern for above flow.

FIGURE 3b. Flow pattern when $k = 5$, $P = 0.442$ (zone B).

FIGURE 3a. Flow pattern when $k = 2.236$, $P = 1$ (zone C).

10. *Numerical results*

The discussion at the close of the previous section indicates the need for precise numerical results concerning those parameter values for which solutions exist.

The University of Minnesota CDC 6600 digital computer was programmed to seek solutions according to the successive approximations scheme described in § 8. As we have already observed following corollary 1, this procedure will converge whenever a solution exists. Numerical solutions were obtained for the fifteen values

$$k^2 = 1, 5, 10, 15, 20, 25, 30, 34.2, 40, 60, 100, 200, 625, 1000, 3600,$$

and for corresponding pressure ratios P starting from $P = 0.2$ and continuing at appropriate increments $\Delta P = 0.002$ until breakdown occurred. Figure 1 has been drawn so as to be in agreement with these calculations. In addition, a successive approximation procedure was carried out for the modified function $G(x)$ defined by (39) with $Q = 0$, in order to obtain a more accurate determination of the curve which separates zone A and zone B.

Figure 2 *a–d* show the result of the numerical calculations in four typical cases, corresponding respectively to parameter values falling in zones C, B (two of these) and A. Stream surfaces† of the accompanying flows are portrayed in figure 3 *a–d*; these graphs illustrate in detail the qualitative behaviour described in the introduction and in chapter II. The stream surfaces in each are drawn for equally spaced values ψ of the streamfunction (see § 1). In consequence, the mass flux of the secondary flow between each of the respective stream surfaces is the same; the particular values of ψ are of course unimportant because of the geometrical similarity of the flow patterns.

The fluid motions shown in figures 3 *b* and *c* involve an inflow near the boundary plane and a downdraft along the vortex axis. This is of course balanced by a corresponding outflow which occurs at an intermediate range of angles: in figure 3 *b* the outflow occurs between 35° and 73° with an asymptotic direction 60°, angles being measured from the horizontal; for the flow in figure 3 *c* the outflow occurs between 13° and 54°, and asymptotically approaches 21°. Because of the interest in motions of this sort, it is valuable to exhibit another example of this type; see figure 4 *a*. The asymptotic direction here is precisely 45°. Isobars of the corresponding pressure field were calculated from equations (3) and (4) and are shown in figure 4 *b*; the pressure decreases monotonically as one radially approaches the origin.

The arrangement of the figures on p. 352 emphasizes the continuity of the flow patterns when k and P are varied. Reading the figures in order 3 *a*, 3 *b*, 4 *a*, 3 *c*, 3 *d*, one sees clearly the smooth transition from upflow (figure 3 *a*) to downflow (figure 3 *d*) through the mechanism of a gradually flattening cascade angle (figures 3 *b*, 4 *a*, 3 *c*).

The particular values of P for the flows shown in figures 3 *b*, 3 *c* and 4 were chosen so that the point (P, k^{-2}) lies on, or nearly on, the right-hand boundary of zone B. Thus the value of P in these cases not only makes the pressure distribution on the boundary plane conform as nearly as possible to the corresponding free vortex flow (see the remarks following the proof of theorem 1), but also results in a minimum axial speed near the vortex line (see theorem 2 (iii)). The related choice $P = 1$ for the flow shown in figure 3 *a* was made for essentially the same reasons.

Now let $\hat{P}(k)$ denote the value of P associated with the right-hand boundary of zone B. Similarly, let $\beta = \beta(k)$ be the asymptotic direction associated with the parameters k, $\hat{P}(k)$, for

† More precisely, their cross-section with planes $\theta = $ constant.

$0 < k^{-2} \leqslant 0.122$. From corollary 1 in § 8, β is the *maximum cascade angle* possible for the given value of k. The calculated graph of $\beta(k)$ is shown in figure 5. When $k^{-2} > 0.122$ the flow corresponding to $P = 1$ can be considered to have a cascade angle (or asymptotic direction) of $90°$. From this point of view we could define $\beta(k) = 90°$ for all $k^{-2} > 0.122$, and continue the graph in figure 5 indefinitely to the right at this constant value. (It also follows from corollary 1 that, for fixed k, the asymptotic direction increases monotonically as P goes from the left- to the right-hand boundary of zone B).
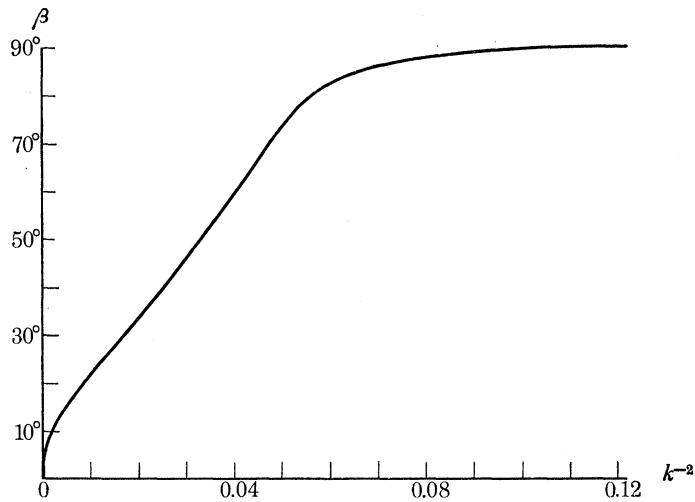
FIGURE 5. Maximum cascade angle β against turbulence level k.

Summary

In the preceding work we have introduced a class of exact solutions of the Navier–Stokes equations, representing the interaction of an infinite vortex line with a plane boundary surface at right angles to the line. As discussed in the introduction, the presence of the boundary surface induces a secondary flow superimposed on the free vortex motion.

Assuming the basic separation of variables given by equations (2), the differential equations governing the motion then show that the flow can be described in terms of a pair of dimensionless parameters k and P. Here P is the ratio of the pressure gradient of the actual flow and of the corresponding free vortex, evaluated on the boundary surface at some fixed radial distance from the vortex line, and k is defined by the ratio $C/2\nu$, where $C = \lim_{r\to 0} r v_\theta$ is the angular momentum of the vortex and ν is the kinematic viscosity. In laminar motion k may be considered as a Reynolds number of the flow; if turbulent motion is assumed then k is more properly a self-regulating parameter indicating the basic level of turbulence present.

It is found that exactly three types of secondary flow régimes can exist. In particular, let us associate to each pair of parameter values k and P a point in the upper half of the (P, k^{-2}) plane, as shown in figure 1. For parameter values falling in zone A the secondary flow is directed downward along, the vortex axis and outward near the boundary surface. For values of k and P falling in zone B the secondary flow again involves a down-draft along the vortex axis, but there is an inflow along the boundary surface. This motion inwards is balanced by a compensating outflow occurring at an intermediate cone of directions around the angle $\alpha_0 = \cos^{-1} a$. Finally, for values of k and P in zone C, the secondary flow consists of an updraft along the vortex axis together with an inflow along the boundary surface. No vortex flow consistent with (2) and the assumed boundary conditions can exist for values of k, P which fall in the remaining zone. It is immediately apparent from the diagram that motions of the third kind, involving an updraft and an inflow, can exist only at low values of k, less than approximately 2.86. Since the values of k appropriate to meteorological phenomena are in many instances apparently somewhat larger than this (see below) it appears that motions of the third kind need not invariably represent the typical case of vortex phenomena, a point to which we shall return later. Typical streamlines for the secondary flows in the three cases above are shown in figures 3 and 4.

The results described in the previous paragraph are found in §4 and in chapter III. As far as the mathematical part of the paper is concerned, the main problem which has been left open is the question of uniqueness, namely whether at most one flow consistent with (2) can exist for a given pair of values k and P. While this seems highly likely, and while the numerical calculations discussed in §10 lead one to the belief that solutions are unique, we have been unable to prove this (on page 927, Goldstik claims that solutions are unique for the case $P = 1$, but his remarks are unsupported). In addition to a uniqueness theorem it would also be useful to know that solutions depend continuously on the parameter values. In view of the complicated nature of the relevant differential equations both this problem and the question of uniqueness may be quite difficult.

Some final remarks may be added concerning the implications of the above results for various meteorological phenomena.† Before proceeding with this, however, it is important to make

---

† Important surveys of the physical and meteorological aspects of tornadoes, dust whirls, and waterspouts have been given by Brooks (1951) and Kessler (1970). Morton (1966) in addition to discussing the behaviour of geophysical vortices also includes a valuable and penetrating discussion of the theoretical aspects of those phenomena. All of these papers have extensive and useful bibliographies.

clear the not inconsiderable problems which attend any such interpretation. We have already argued in the introduction that buoyancy and compressibility effects should not be of major importance in tornado phenomena, for the speeds are apparently neither so high that compressibility will be important nor so low that buoyancy can exert an immediate effect.† More significant is the idealization of a tornado core by a line vortex, for in a real fluid the innermost parts of a vortex surely will not enjoy the inverse distance law of speed which governs a line vortex. Whether such an idealization is justified is, at our present state of knowledge of viscous or turbulent vortex cores, a matter which must remain somewhat intuitive, resting ultimately on the same conviction which justifies the study of line vortices in ideal fluids—the belief that they are reasonably accurate asymptotic models of the physical world. Following this line of thought, one could argue that the velocity singularity at the vortex line (and the corresponding pressure singularity) might reasonably be smoothed out by a matching process analogous to the Rankine combined vortex of ideal fluid theory. While such a process is to be desired, and might produce important further insight into the structure of tornado cores, nevertheless lacking such a desideratum one tends to accept the (admittedly meagre) experimental measurements in tornadoes which point to the validity of the inverse distance law of speed except in the neighbourhood of the core itself (see particularly Lewis & Perkins 1953) and to accept the intuitive picture (Morton 1966) of a tornado or waterspout as a strongly concentrated core of vorticity embedded in a weakly vortical environment, for which the circulation in circular paths outside the core is relatively constant.

In modelling large-scale meteorological phenomena one must assume the flow to be turbulent if any progress is to be made. Following the semi-empirical approach of Boussinesq we thus consider ν to be a virtual or eddy viscosity, a self-regulating parameter of the flow. There is of course no reason to expect a precise value for ν nor in fact is one necessary; on the other hand, numerical and empirical evidence given by Schlichting (1960, chapter 23) indicates that a reasonably accurate hypothesis is

$$\nu \approx \sigma qr,$$

where q is the flow speed, r the radial distance from the axis of the vortex, and σ a dimensionless factor in the range 0.1 to 0.2. By (2), (6), (7) the quantity qr is constant along radii through the origin, takes the value 0 at the boundary plane, and tends to C as one approaches the vortex axis. One may then put $\nu \approx \frac{3}{4}\sigma C$ throughout the flow field with fair approximation.‡ Thus we are led to consider values of the parameter $k = C/2\nu$ in the approximate range

$$k = 3 \quad \text{to} \quad k = 7.$$

Naturally, rather than taking this *a priori* point of view we could equally well consider k as a parameter to be determined by observational evidence.

With k fixed, numerical values of P presumably should be as near to 1 as possible in order for the pressure distribution to follow most nearly that in a free vortex (see the remarks following theorem 1). For values of k between 3 and 7, this places the parameter point (P, k^{-2}) on the curved part of the right-hand boundary of zone B; that is, $P = \hat{P}(k)$ where $\hat{P}(k)$ was defined in the previous section. The associated motions exhibit an inflow along the boundary plane and a

---

† Vonnegut has argued that electrical activity may be important in certain tornado phenomena; in such cases the present analysis would evidently be insufficient.

‡ Another procedure, computationally and theoretically more difficult, would be to set $\nu = \sigma(\alpha) C$ where $\sigma(\alpha)$ is some function deliberately chosen to reflect a dependence of the turbulence level on the angle α. This approach would also seem to be in agreement with Turner's view (1966, p. 400) of self-regulating turbulence in a tornado.

descending motion along the axis. The corresponding cascade angle as determined from figure 5 varies from 89° when $k = 3$ to 33° when $k = 7$. Figures 3b and 4a illustrate two flows from this range of parameter values, those for $k = 5$ and $k = 5.85$. The associated pressure fields are also of interest, since the visible funnel of a tornado or waterspout is caused by the condensation of water vapour when the dynamic pressure falls below the local vapour pressure (Brooks 1951). Consequently the outline of a funnel cloud should approximately represent an isobar of the motion. The general appearance of the equipressure lines in figure 4b, and especially their gradual widening at the top, agrees in this respect with the typical configurations of tornado and waterspout funnels.

There is considerable observational evidence concerning the form of dust and mist cascades at the foot of tornadoes and waterspouts, both descriptive and in the form of photographs taken by persons who had, so to speak, the presence of mind to reach for their cameras rather than run for their cellars.

Figure 6a, plate 2, is a historic photograph of a waterspout off Cottage City, Martha's Vineyard, 19 August 1896. Though the photograph has suffered from poor development especially at the critical point where the waterspout meets the horizon, one may nevertheless estimate the cascade angle to be about 40° to 45°. The straight funnel and almost stationary aspect of this waterspout are also particularly remarkable. Figure 6b, plate 2, shows a tornado at Elbow Lake, Minnesota. The cascade angle averages about 40°; in this case the inclination of the vortex axis explains the fact that the angle is different on the two sides of the tornado. Figure 6c, plate 3, shows a similar example, this one from Kansas (the ubiquitous telephone pole dominates the scene as usual). Figure 6d, plate 3, is a particularly good picture of the destructive tornado of 11 April 1965 which struck Elkhart, Indiana. Althought the dust funnel is more fully developed here than in the previous pictures, we again see a strong conical outflow, the average angle of the right hand debris cone being about 48°. Figure 6e, plate 4, is a beautiful photograph of a tornado whose cascade angle is either 90° (that is, upflow throughout the vortex as in mode C) or very nearly 90°. A number of further photographs can be found on pages 24, 88, and 136 of Flora's monograph (1954).

If air flow in a tornado is assumed to follow the pattern associated with zone B, and if dust or debris accurately traces the motion, then material swept up from the area near the foot of the vortex would become concentrated on the conical stream surface defined by the angle β and would appear to the observer as a dark region with a conical lower boundary, beneath which the air is clear. *Therefore to the extent that debris (or mist) in figures 6a to 6d faithfully traces the air motion, we obtain specific observational evidence in favour of mode B, and of the associated occurrence of descending air motion in the core of tornadoes whose foot reaches to the ground.* As Professor Lighthill has noted in a private communication, the main problem in identifying the debris shower with the air motion is that debris is subject to gravity and centrifugal force in addition to the swirling motion of the air. Gravitational and inertial effects on dust are doubtless slight over the time scales involved and consequently can be ignored. Centrifugal effects, however, are certainly larger, but whether they can account for cascades in a flow with a central updraft is doubtful. There are a number of reasons for this, but perhaps the strongest is simply that the lower boundaries of observed cascades are relatively straight and well defined. Assuming that such cascades of dusty debris were in fact caused by centrifugal effects in an air motion otherwise involving a simple inflow and updraft, the vertical component of the air velocity would have to fall off like the cube of the radius, which appears unlikely. The random distribution of size among the dust or debris particles would also

dictate a variety of trajectories and accordingly a badly defined cascade boundary, accentuated further by what would seem to be an unstable moving front (Taylor instability). The appearance of the cascade itself thus argues against centrifugal force as the generating mechanism, though naturally this feature will have some moderating effect.† The author has seen motion pictures in which large pieces of debris were swirled out at approximately 45° while the dust cascade stood at about 60°. If centrifugal force were the prime mechanism one would expect a very much larger relative effect on the heavier debris; it would thus seem more reasonable to view the dust as a tracer of the air flow and the larger debris alone as significantly affected by centrifugal force.

An extreme example of a dust cloud is shown in figure 6f, plate 4. The tornado was 11 000 feet distant and was moving at a speed of about 15 miles/h to the left. It is difficult to explain the immense size of the cloud by centrifugal force; moreover, were there a significant central updraft it is hard to imagine that the funnel could have remained unobscured by debris. One may conjecture that the dust cloud is the result, therefore, of a prolonged cascade effect at a fairly low angle. If we estimate this intuitively from the figure to be about 30°–35°, the corresponding k is about 7, indicating a relatively low level of turbulence. Significantly this tornado not only moved slowly but had a high degree of coherence since it remained actively in contact with the ground and seriously destructive for some $1\frac{1}{2}$ hours.

The possibility of central downflow in a tornado is reinforced by the careful observational evidence of Hoecker (1960), who writes, 'A sequence of photographs of [a] tornado on file [shows] the ground-based debris cloud increasing in diameter as the suspended funnel widened and lowered toward the ground. No debris was seen to rise in the region beneath the cut-off tip of the condensation funnel but all of it ascended exterior to the tapered cylindrical funnel which widened upward. In the pictures and movies of the Dallas tornado no large clouds of dust or spray, or chunks of structures were observed ascending along the trunk when the trunk was touching the ground. Had any large chunks been carried upward inside the funnel, surely a few would have been thrown outward through the funnel wall by centrifugal force much as they were observed to do when the funnel tip had retracted.' The problem of descending flow has also been logically considered by Morton (1966). He concludes that 'The radial flow at ground level is inwards towards the low-pressure core, and hence the net vertical flow must be upward; however, this does not imply that the flow need be upward over the whole core section, and it is quite possible that under appropriate circumstances there may be downflow near the axis'. Following a consideration of Hoecker's work, he states further, 'When the funnel descends to the ground the column of dust and debris often forms a shell which is clearly separated from the funnel within. Such a shell might be formed by particles swept up between an outer flow spiralling in and up, and an inner flow spiralling down the axis and then out and up to form a cellular core; and this seems more plausible than the traditional explanation that the dust column is formed by centrifugal separation of material raised in a strong upflow near the funnel axis, which explains neither the frequent separation between shell and inner core nor the obviously cylindrical shell-form sometimes observed.'

Some direct observations of downflow have also been reported. E. M. Brooks, in the *Compendium of meteorology*, states that 'whirling air coming down at 1800 ft/min has actually been encountered by a glider in a desert dust whirl', and M. Hale, in a private communication to the author, has described the flight of a naval airplane through a mild waterspout, in which the plane

† The fact that figure 6e shows no cascade is itself not an argument against centrifugal force, since the dust carried aloft may be relatively fine and dry.
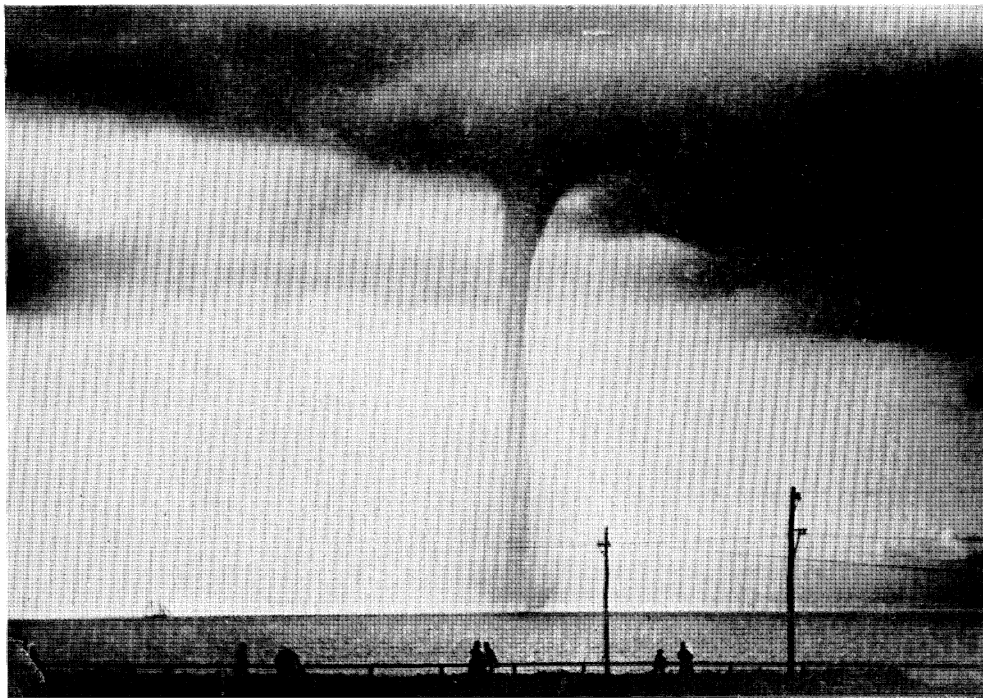
FIGURE 6*a*. Waterspout off Martha's Vineyard, 19 August 1896. (Courtesy Dukes County Historical Society, Edgartown, Mass.)

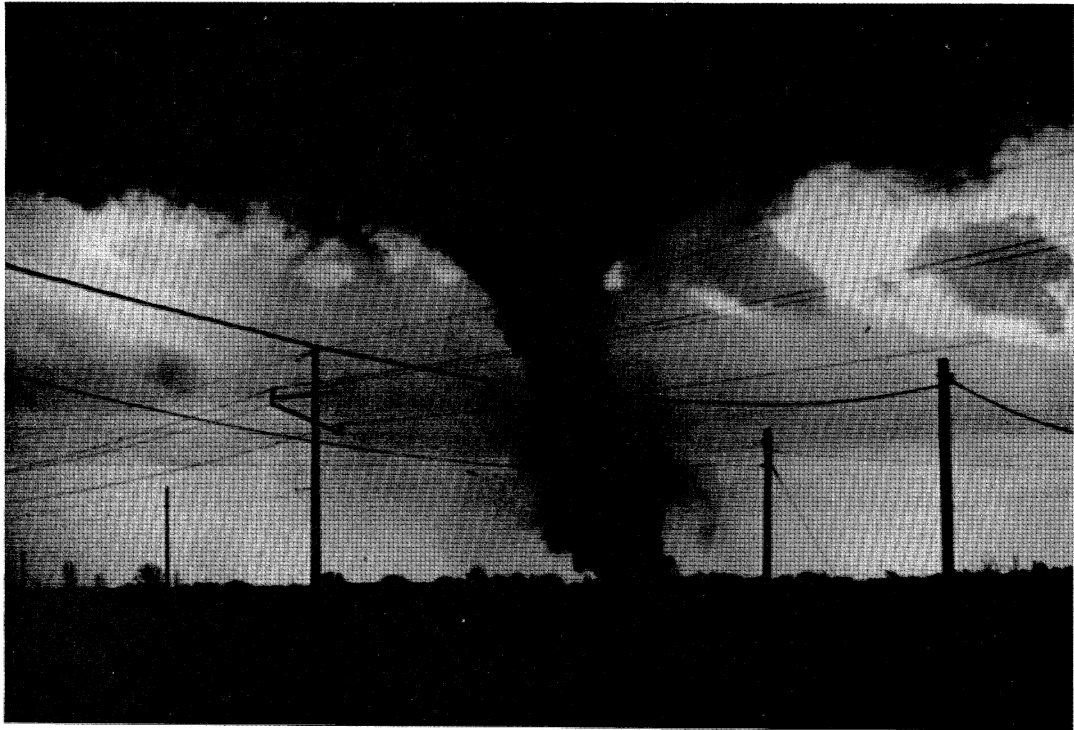FIGURE 6*b*. Tornado at Elbow Lake, Minnesota, 5 September 1969. (Photograph by Olaf Dybdal.)

FIGURE 6c. Tornado in Kansas, 21 April 1967.
FIGURE 6d. Tornado at Elkhart, Indiana, 11 April 1965. (Photograph by Paul Huffman.)

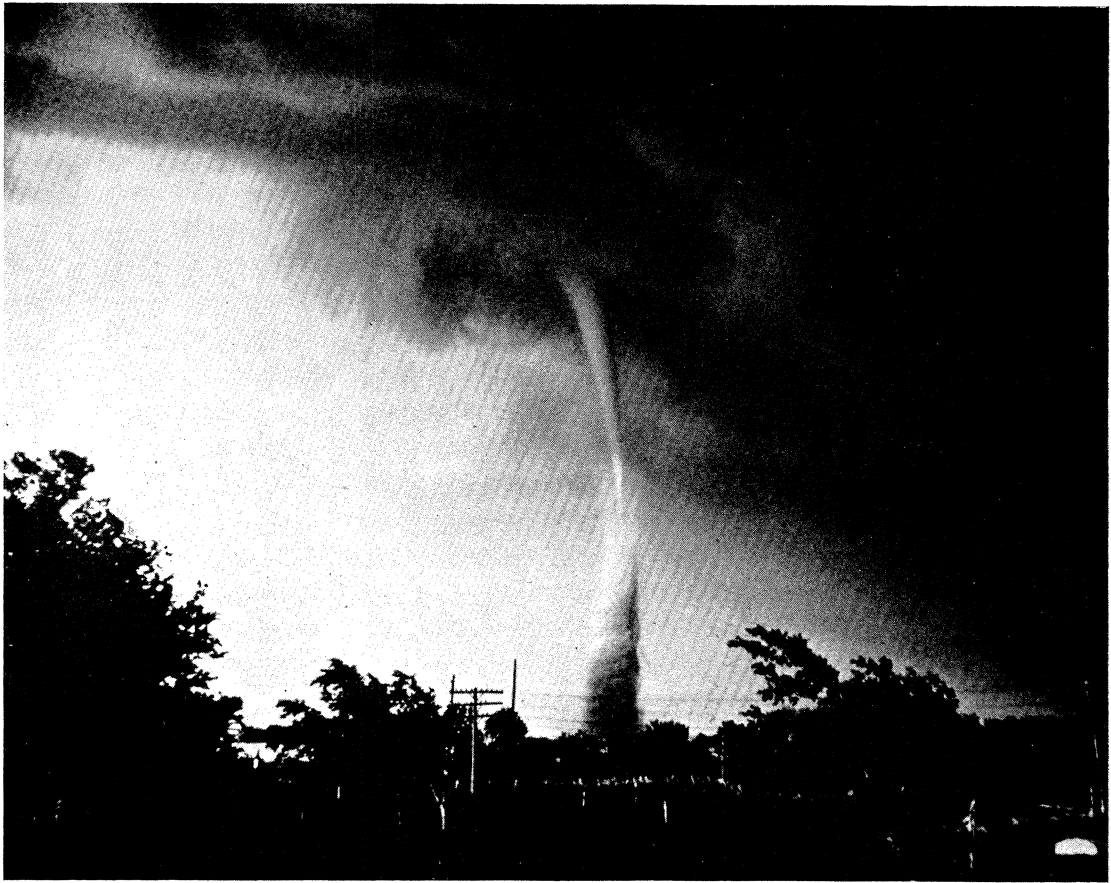FIGURE 6*e*. Tornado at Enid, Oklahoma, 5 June 1966. (Environmental Science Services
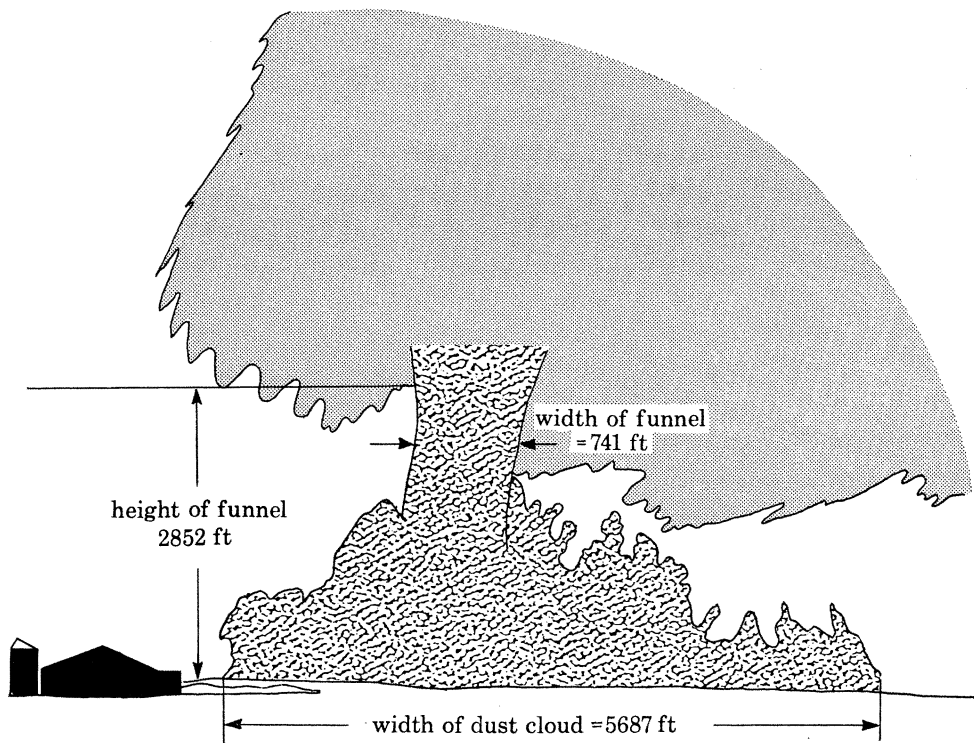Administration photograph by Leo Ainsworth.)

FIGURE 6*f*. Tornado at Scottsbluff, Nebraska, 27 June 1955. (Drawing made from
measured photograph, Van Tassel 1955, p. 257.)

was seen to experience a downward turn. In addition to direct reports of this kind, a series of interesting experiments has been conducted by Ward (1956). In laboratory-generated vortices maintained by a relative low-pressure region above the vortex location, Ward has observed combined inflow at the lower boundary and descending air motion in the core at certain critical combinations of the flow parameters when the motion becomes turbulent. The theoretical work of Sullivan (1959) and Bellamy-Knights (1970) likewise admits the possibility of central down-drafts in vortex cores, though as we have remarked in the introduction their models do not effectively reflect either the adherence condition at the boundary surface or the fact that the velocity should approach zero at large radial distances. Kuo (1967) has determined approximate numerical solutions of the equations representing vortex motion in an unstably stratified atmo-sphere, and finds again the possibility of central downflow. Emphasizing the possible complexity of the flow situation, Turner (1966) has presented approximate solutions of the Navier–Stokes equations with annular downdrafts and a central updraft. His work, like that of Sullivan, assumes solid rotation at large radial distances.

While the preceding discussion has focused particularly on the observational evidence for flow patterns of type B, this was held necessary in view of the widely disseminated opinion that tornadoes and waterspouts are invariably updraft phenomena. It should not be concluded from this particular emphasis, however, that updraft phenomena on the other hand are unlikely or unusual. Besides the observational evidence of figure 6e, plate 4, and other similar photographs, Hoecker (1960) specifically notes that 'the air flow pattern in a tornado is complex and changes from one tornado to another and may change from time to time in any one tornado'. Similarly, Morton says 'It may be noted that there is no reason why tornado vortices should all be dynami-cally similar.' To the extent that the present model represents tornado phenomena, it bears out both statements, since for different values of k the resulting motions are *not* dynamically similar, nor is it necessary that k retain a constant value through the life of a given tornado. On the other hand, one qualitative attribute of the model remains valid irrespective of the choice of k, provided only that the parameter P is determined by the physically motivated relation $P = \hat{P}(k)$. Indeed, the corresponding flows are then of type B if $k > 2.86$ and of type C with $P = 1$ when $k \geqslant 2.86$: *in either case the motion near the boundary plane is inwards toward the vortex axis*. Observational evidence (Brooks 1951) on this point is convincingly in agreement.

Although the conditions $k < 2.86$, $P = 1$ are consistent with observational evidence (see particularly figure 6e) and with the prevalent inflow-updraft theory, nevertheless, we have earlier concluded that k would tend to fall in the range 3 to 7. The difficulty here is, however, more apparent than real. One must not, to begin with, place excessive confidence in empirical values of ν carried over from one situation to another, so that we need not automatically eliminate values of k which remain slightly less than 3. Alternately, when $k = 3$ the cascade angle is approximately 89°, and within any reasonable limits of accuracy this amounts to a motion with a central updraft. Finally, the various reservations in the basic model which we have discussed earlier must surely be taken into consideration in applying the conclusions.

In summary, beyond the details of numerical computation and the idealization of the modelling process, the present discussion indicates the possibility of a complete analysis of the steady state interaction of a line vortex and a boundary plane in a viscous fluid. The family of solutions which has been obtained moreover displays a number of the diverse effects observed in both tornadoes and waterspouts, and places particular emphasis on the possibility of centrally descending air motion as a realistic flow pattern in geophysical vortex phenomena.

References

Bellamy-Knights, P. G. 1970 An unsteady two-cell vortex solution of the Navier–Stokes equations. *J. Fluid Mech.* **41**, 673–687.

Berker, R. 1963 Intégration des équations du mouvement d'un fluide visqueux incompressible. *Hand. Phys.* VIII/2. Berlin: Springer-Verlag.

Burgers, J. M. 1948 A mathematical model illustrating the theory of turbulence. *Adv. appl. Mech.* **1**, 197–199.

Burggraf, O., Stewartson, K. & Belcher, R. 1971 Boundary layer induced by a potential vortex. *Phys. Fluids.* **14**, 1821–1833.

Coddington, E. & Levinson, N. 1955 *Ordinary differential equations.* New York: McGraw-Hill.

Dunford, N. & Schwartz, J. 1958 *Linear operators*, **1**. New York: Interscience.

Goldstik, M. A. 1960 A paradoxical solution of the Navier–Stokes equations. *Prik. Mat. Mekh.* **24**, 610–621. Translated in *J. appl. Math. Mech.* **24**, 913–929.

Long, R. R. 1958 Vortex motion in a viscous fluid. *J. Meteor.* **15**, 108–112.

Long, R. R. 1961 A vortex in an infinite viscous fluid. *J. Fluid Mech.* **11**, 611–624.

Rott, N. 1958 On the viscous core of a line vortex. *Z.A.M.P.* **9**b, 543–553.

Schlichting, H. 1960 *Boundary layer theory*, Fourth ed. New York: McGraw-Hill.

Sullivan, R. D. 1959 A two-cell solution of the Navier–Stokes equations. *J. Aero/Space Sci.* **26**, 767–768.

Meteorological references

Brooks, E. M. 1951 Tornadoes and related phenomena. *Compendium of meteorology.* Boston: Amer. Met. Society.

Flora, S. D. 1954 *Tornadoes in the United States.* University of Oklahoma Press, Norman.

Hoecker, W. H. 1960 Wind speed and air flow patterns in the Dallas tornado of April 2, 1957. *Mon. Weath. Rev.* **88**, 167–180.

Kessler, E. 1970 Tornadoes. *Bull. Am. Met. Soc.* **51**, 926–936.

Kuo, H. L. 1967 Note on the similarity solutions of the vortex equations in an unstably stratified atmosphere. *J. atmos. Sci.* **24**, 95–97. Cf. also *J. atmos. Sci.* **23**, 25–42 (1966).

Kuo, H. L. 1971 Axisymmetric flows in the boundary layer of a maintained vortex. *J. atmos. Sci.* **28**, 20–41.

Lewis, W. & Perkins, P. J. 1953 Recorded pressure distribution in the outer portion of a tornado vortex. *Mon. Weath. Rev.* **81**, 379–385.

Morton, B. R. 1966 Geophysical vortices. *Prog. aeronaut. Sci.* **7**, 145–193.

Oerstad, H. C. 1839 On water-spouts. *Am. J. Sci. Arts* **37**, 250–267. Translated from *Schumacher's Jahrbuch für* 1838.

Turner, J. S. 1966 The constraints imposed on tornado-like vortices by the top and bottom boundary conditions. *J. Fluid Mech.* **25**, 377–400.

Van Tassel, E. L. 1955 The North Platte Valley tornado outbreak of June 27, 1955. *Mon. Weath. Rev.* **83**, 255–264.

Vonnegut, B. 1960 Electrical theory of tornadoes. *J. geophys. Res.* **65**, 203–212.

Ward, N. B. 1956 Temperature inversion as a factor in the formation of tornadoes. *Bull. Am. Met. Soc.* **37**, 145–151. Cf. also *Nat. Sev. Storms Lab. Tech. Memo.* no. 52 (1970).

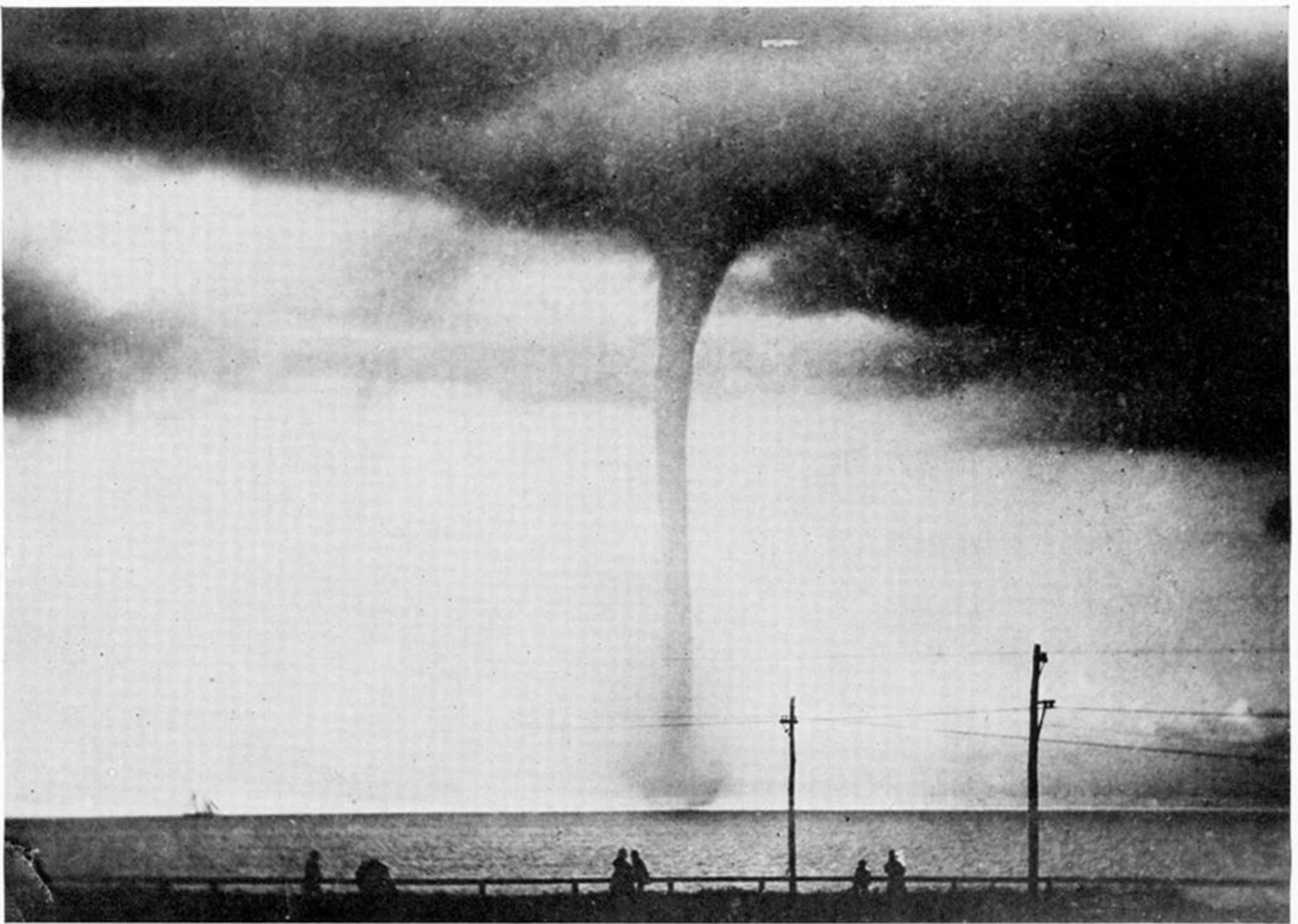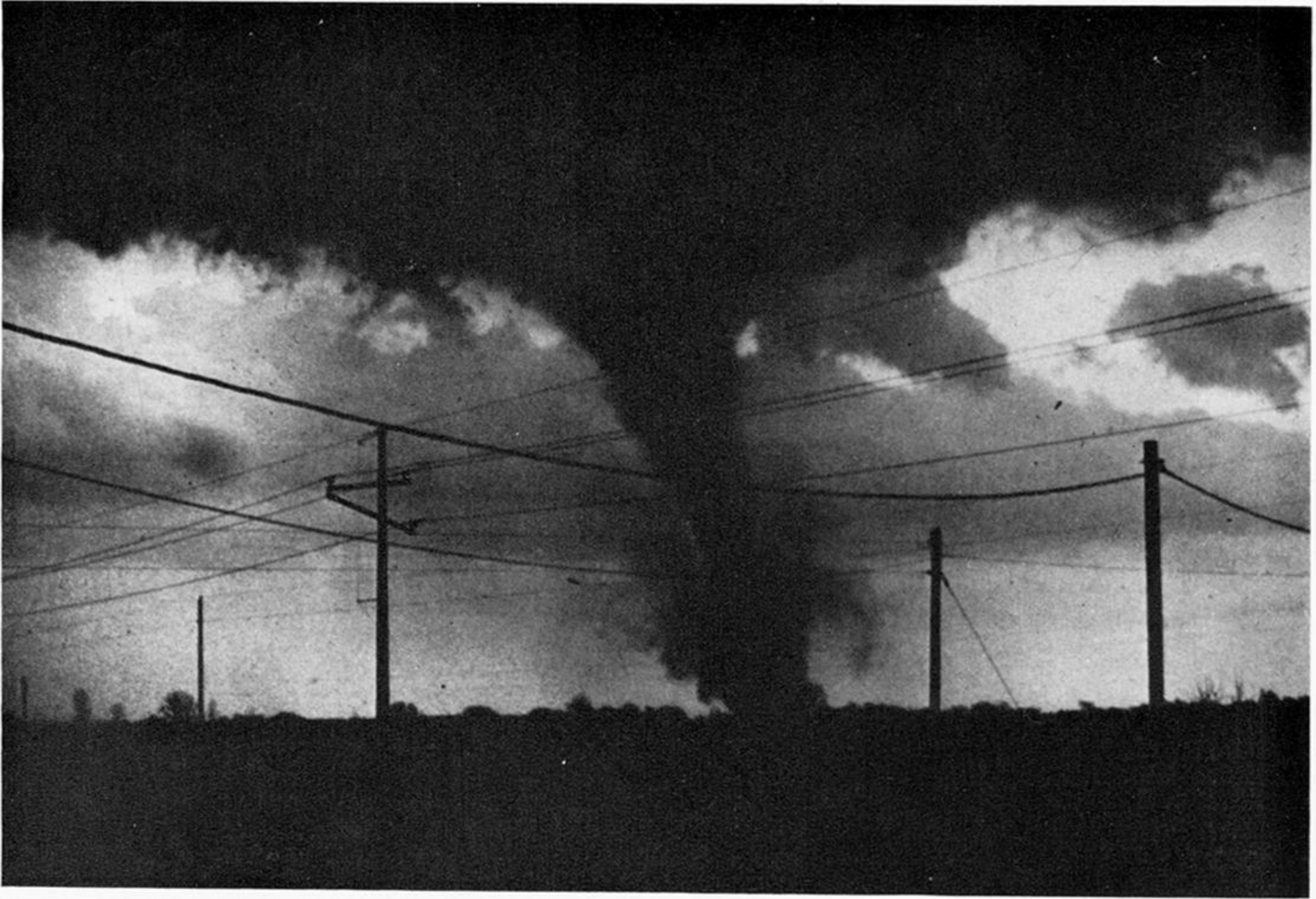Ying, S. J. & Chang, C. C. 1970 Exploratory model study of tornado-like vortex dynamics. *J. atmos. Sci.* **27**, 3–14.

FIGURE 6a. Waterspout off Martha's Vineyard, 19 August 1896. (Courtesy Dukes County Historical Society, Edgartown, Mass.)

FIGURE 6b. Tornado at Elbow Lake, Minnesota, 5 September 1969. (Photograph by Olaf Dybdal.)

FIGURE 6c. Tornado in Kansas, 21 April 1967.

FIGURE 6d. Tornado at Elkhart, Indiana, 11 April 1965. (Photograph by Paul Huffman.)

FIGURE 6e. Tornado at Enid, Oklahoma, 5 June 1966. (Environmental Science Services
Administration photograph by Leo Ainsworth.)